

Weak Hopf Algebras II. Representation Theory, Dimensions, and the Markov Trace

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Abstract

If A is a weak C^* -Hopf algebra then the category of finite dimensional unitary representations of A is a monoidal C^* -category with monoidal unit being the GNS representation D_ε associated to the counit ε . This category has isomorphic left dual and right dual objects which leads, as usual, to the notion of dimension function. However, if ε is not pure the dimension function is matrix valued with rows and columns labelled by the irreducibles contained in D_ε . This happens precisely when the inclusions $A^L \subset A$ and $A^R \subset A$ are not connected. Still there exists a trace on A which is the Markov trace for both inclusions. We derive two numerical invariants for each C^* -WHA of trivial hypercenter. These are the common indices I and δ , of the Haar, respectively Markov conditional expectations of either one of the inclusions $A^{L/R} \subset A$ and $\hat{A}^{L/R} \subset \hat{A}$. In generic cases $I > \delta$. In the special case of weak Kac algebras we show that $I = \delta$ is an integer.

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1 Introduction

We continue the analysis of weak Hopf algebras started in [4] the main issue now being the structure of weak C^* -Hopf algebras. We use the notations and terminology of [4] which will be referred to as I. and the theorems, equations, etc. there will be quoted as (I.3.12) for example.

Being a "quantum groupoid", i.e. a generalized concept of symmetry, weak Hopf algebras (WHA's) have representation categories with monoidal product and notions of left dual and right dual objects. In case of C^* -WHA's this category $\mathbf{rep} A$ is a monoidal C^* -category in which the left dual and right dual are canonically isomorphic, due to the existence of the canonical grouplike element g of Prop.I.4.9. $\mathbf{rep} A$ is semisimple and the finite set $\mathbf{Sec} A$ of equivalence classes of irreducibles are called the set of *sectors*,

a term borrowed from quantum field theory. The subset $\mathcal{V}ac A$ of sectors that occur in the decomposition of the monoidal unit V_ε of $\mathbf{rep} A$ are called *vacua*. This name is supported by the behaviour of general sectors under the monoidal product: They have a groupoidlike composition law in which $\mathcal{V}ac A$ plays the role of the set of units. Thus generic sectors can be thought of as interpolating between different vacua, we call them *solitons*, again by some, however vague, quantum field theoretic motivation. (For an approach to solitons in algebraic quantum field theory see [8].)

As it is well known isomorphism of the left dual and right dual allows one to introduce a faithful tracial map $\phi_V: \text{End } V \rightarrow \text{End } V_\varepsilon$ for each object V of $\mathbf{rep} A$ which leads then to a notion of dimension d_V of representations. For uniqueness of ϕ_V and therefore of d_V one uses a distinguished choice of the rigidity intertwiners inherent in the definition of duals. If the WHA A is pure, i.e. its trivial representation V_ε is irreducible, then this choice is precisely the *standard* rigidity intertwiners of [11]. If V_ε is not irreducible, i.e. decomposes into more than one vacuum representation, then standardness needs a modification which results in a notion of dimension which assigns to the representation V a matrix \mathbf{d}_V the rows and columns of which are labelled by the set of vacua. Irreducibles $q \in \mathcal{S}ec A$ have dimension matrices \mathbf{d}_q which are products of a matrix unit with a positive number d_q , sometimes also called the dimension of q . The matrix unit content of \mathbf{d}_q is, however, necessary for the dimension function $V \mapsto \mathbf{d}_V$ to be multiplicative and additive.

Sections 2 and 3 are dealing with the structure of representation categories of WHA's, with soliton sectors, and the dimension matrix. As a little deviation from the main course, in Subsection 3.8 we construct Frobenius-Schur indicators for C^* -WHA's that has already been introduced in [7].

There is an other aspect of WHA's that go well beyond their representation categories. It is the 2-dimensional array of inclusions one obtains from the two inclusions $A^L \subset A \supset A^R$ by repeated applications of the Jones construction. This is a kind of standard invariant [17] for a to-be-constructed depth 2 inclusion of algebras for which the tower $A^L \subset A \subset A \rtimes \hat{A} \subset \dots$ is the (first) derived tower. In the C^* setting this offers a way to describe finite index depth 2 inclusions of von Neumann algebras (of finite dimensional centers) as a crossed product w.r.t an action of a C^* -WHA [15]. This is a special case of the much more general situation considered in [6].

The above mentioned array of inclusions (see Fig.2) can be thought of as the selfintertwiner algebras of certain 1-morphisms in a C^* -2-category with duals for 1-morphisms. Although this 2-category will not be made precise in this paper, it offers a good intuitive guideline to describe the structure of WHA's algebraically.

For example, we find an extension of the dimension function to the sectors of $A^{L/R}$, which would be meaningless in the representation category of $A^{L/R}$ since they are not coalgebras. The dimension \mathbf{d}_a of a sector $a \in \mathcal{S}ec A^L$ is again a matrix but with rows from $\mathcal{V}ac A$ and columns from $\mathcal{V}ac \hat{A}$. By additivity, the dimension matrix of $A^L \cong \oplus_a M_{n_a}$ (more precisely of the 1-morphism the selfintertwiner algebra of which is A^L) is given by $\mathbf{d}^L = \sum_a n_a \mathbf{d}_a$ and it plays the role of a generator. The dimension matrix \mathbf{d}^R of A^R (i.e. that of the 1-morphism dual to that of A^L) is the transpose of \mathbf{d}^L and those of the WHA's A and \hat{A} are obtained as the matrix products

$$\mathbf{d}_A = \mathbf{d}^L \mathbf{d}^R, \quad \mathbf{d}_{\hat{A}} = \mathbf{d}^R \mathbf{d}^L. \quad (1.1)$$

The dimension matrices \mathbf{d}_A and $\mathbf{d}_{\hat{A}}$ are of course the same that one obtains from their representation categories, as C^* -WHA's. In case of a finite group these relations become

unduly trivial: They simply say that A and \hat{A} have the same dimension and both of them are the squares of their square roots.

Not every triple $A^L \subset A \supset A^R$ can become the left and right subalgebra of a C^* -WHA A . In order to understand what restrictions this imposes on the given inclusion triple measure theoretic concepts, such as the Haar conditional expectations $E^{L/R}: A \rightarrow A^{L/R}$ and the Markov conditional expectations $E_M^{L/R}: A \rightarrow A^{L/R}$, turn out to be useful. We prove in Theorem 4.4 that a common Markov trace on A exists for the two inclusions $A^L \subset A$ and $A^R \subset A$ implying, among others, that the inclusion matrices of all of the connected components of $A^L \subset A$ have the same norm. Moreover this norm and the analogue norm for \hat{A} coincide, although the inclusion matrix of $\hat{A}^L \subset \hat{A}$ and that of $A^L \subset A$ may be completely different.

The indices I and δ of the Haar and the Markov conditional expectations, respectively, provide "scalar" (more precisely hypercentral) elements of A . They can be expressed algebraically in terms of the integer *multiplicities* n_q and the intrinsic *dimensions* d_q only in special cases. We give these special cases here:

$$I = \dim A^L \cdot \sum_q d_q^2 \quad \text{if } A \text{ is pure and } S^2|_{A^L} = \text{id}|_{A^L} \quad (1.2)$$

$$\delta = \sum_q n_q d_q \quad \text{if } A \text{ is pure.} \quad (1.3)$$

For the general case see Subsections 4.3 and 4.4. All these formulae generalize the well known identity $\dim A = \sum_q n_q^2$ valid for a finite group or a finite dimensional C^* -Hopf algebra. The occurrence of two different indices (in general $I \geq \delta$) is related to non-triviality of S^2 , the square of the antipode. In weak Kac algebras we show that $I = \delta$ and it is always an integer. In case of pure weak Kac algebras this integer is nothing but $\dim A / \dim A^L$, suggesting that pure weak Kac algebras might be very close to what has been called the blowing up of (quasi-)Hopf algebras in [3].

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2 Representations of WHA's

For A an associative algebra over the field K let $\mathbf{mod} A$ denote the category of finite dimensional left A -modules. Therefore the objects of $\mathbf{mod} A$ are the finite dimensional vector spaces V equipped with an action $A \ni a, V \ni v \mapsto a \cdot v \in V$ of A which is nondegenerate: $1 \cdot v = v$. Sometimes it will be convenient to use the algebra homomorphism $D_V: A \rightarrow \text{End}_K V$, the representation on V , i.e. $D_V(a)v := a \cdot v$. The space of intertwiners (or arrows) from the object V to the object W are denoted by $\text{Hom}(V, W)$ and consists of K -linear maps $T: V \rightarrow W$ satisfying the intertwiner property $T(a \cdot v) = a \cdot T(v)$, $a \in A, v \in V$. The composition of arrows $T_1 \in \text{Hom}(V, W)$ and $T_2 \in \text{Hom}(U, V)$ are denoted by $T_1 \circ T_2$. The unit arrow at the object V is $D_V(1)$ and will be denoted by 1_V .

In this section we will investigate the additional structure $\mathbf{mod} A$ acquires by A having a weak Hopf structure.

2.1 Monoidal structure

The coproduct Δ allows us to define a *monoidal product* of left A modules and their intertwiners. At first one chooses a strictly monoidal tensor product \otimes in the category $\mathbf{vec} K$ of finite dimensional vector spaces over K . Then for two objects V and W in $\mathbf{mod} A$ one makes the tensor product $V \otimes W$ into a left A -module by setting $a \cdot (v \otimes w) := x_{(1)} \cdot v \otimes x_{(2)} \cdot w$. Since this module is degenerate in general, the monoidal product $V \times W$ in $\mathbf{mod} A$ is defined as the submodule $\Delta(1) \cdot (V \otimes W)$. For intertwiners $T_i \in \text{Hom}(V_i, W_i)$, $i = 1, 2$, the monoidal product $T_1 \times T_2 \in \text{Hom}(V_1 \times V_2, W_1 \times W_2)$ is simply the restriction of $T_1 \otimes T_2$ onto the subspace $V_1 \times V_2 \subset V_1 \otimes V_2$. Coassociativity of Δ and strict monoidality of \otimes immediately imply

$$(T \times R) \times S = T \times (R \times S) \quad (2.4)$$

$$(T \times R) \circ (S \times U) = (T \circ S) \times (R \circ U) \quad (2.5)$$

$$\mathbf{1}_V \times \mathbf{1}_W = \mathbf{1}_{V \times W} . \quad (2.6)$$

Although these properties are those of a strict monoidal category, we cannot expect $(\mathbf{mod} A, \times)$ to be strictly monoidal since the monoidal unit for \otimes (some 1-dimensional vector space) may not belong to $\mathbf{mod} A$. A natural candidate for the *unit object* (or *monoidal unit*) is the trivial representation V_ε defined in Definition I.2.13. In the sense of relaxed monoidal categories (see [12]) V_ε is a monoidal unit if there exist invertible arrows $U_V^L \in \text{Hom}(V, V_\varepsilon \times V)$, $U_V^R \in \text{Hom}(V, V \times V_\varepsilon)$, for each A -module V such that they are natural in V ,

$$\left. \begin{aligned} (\mathbf{1}_{V_\varepsilon} \times T) \circ U_V^L &= U_W^L \circ T \\ (T \times \mathbf{1}_{V_\varepsilon}) \circ U_V^R &= U_W^R \circ T \end{aligned} \right\} \quad T \in \text{Hom}(V, W) , \quad (2.7)$$

and satisfy the triangle identities

$$U_V^L \times \mathbf{1}_W = U_{V \times W}^L \quad (2.8)$$

$$\mathbf{1}_V \times U_W^L = U_V^L \times \mathbf{1}_W \quad (2.9)$$

$$\mathbf{1}_V \times U_W^R = U_{V \times W}^R \quad (2.10)$$

for all objects V and W .

Proposition 2.1 *The trivial left A -module $V_\varepsilon = \hat{A}^R$ together with the maps*

$$U_V^L: v \mapsto 1_{(1)} \rightarrow \hat{1} \otimes 1_{(2)} \cdot v \in V_\varepsilon \times V \quad (2.11)$$

$$U_V^R: v \mapsto 1_{(1)} \cdot v \otimes 1_{(2)} \rightarrow \hat{1} \in V \times V_\varepsilon \quad (2.12)$$

is a unit object of $(\mathbf{mod} A, \times)$.

Proof: The arrows U_V^L and U_V^R are invertible arrows with inverses

$$U_V'^L: V_\varepsilon \times V \rightarrow V , \quad U_V'^L(\varphi^R \otimes v) = (1 \leftarrow \varphi^R) \cdot v \quad (2.13a)$$

$$U_V'^R: V \times V_\varepsilon \rightarrow V , \quad U_V'^R(v \otimes \varphi^R) = (\varphi^R \rightarrow 1) \cdot v , \quad (2.13b)$$

respectively. Indeed, one can easily check that $U_V'^L \circ U_V^L = \mathbf{1}_V$, $U_V^L \circ U_V'^L = \mathbf{1}_{V_\varepsilon \times V}$ and similar expressions for the right U -arrows. The triangle identities in turn follow from the \hat{A} -versions of axioms (A.7a–b) and Eqns (I.2.11a–b). The details of the calculation are omitted. For more about the monoidal structure we refer to [14]. *Q.e.d.*

2.2 Left duals and right duals

In this subsection we construct left and right dual objects in $\mathbf{mod} A$ using the antipode S .

The dual space $\hat{V} := \text{Hom}_K(V, K)$ of a left A -module V is canonically a right A -module: $\langle f \cdot x, v \rangle := \langle f, x \cdot v \rangle$, $f \in \hat{V}, x \in A, v \in V$. In order to make it a left A -module we can use either one of the antialgebra maps S or S^{-1} . So the *left dual* module \bar{V} is defined to be the dual space \hat{V} with left A -action $x \cdot f := f \cdot S(x)$ and the *right dual* \vec{V} is the same space equipped with the action $x \cdot f := f \cdot S^{-1}(x)$.

In order to establish $V \mapsto \bar{V}$ and $V \mapsto \vec{V}$ as the object maps of a left duality functor and a right duality functor, respectively, we introduce the *left and right rigidity intertwiners*

$$\bar{R}'_V: \bar{V} \otimes V \rightarrow V_\varepsilon, \quad f \otimes v \mapsto f(1_{(1)} \cdot v)1_{(2)} \rightarrow \hat{1} \quad (2.14a)$$

$$\bar{R}_V: V_\varepsilon \rightarrow V \otimes \bar{V}, \quad \varphi^R \mapsto \sum_i (1 \leftarrow \varphi^R) \cdot v_i \otimes f^i \quad (2.14b)$$

$$\vec{R}'_V: V \otimes \vec{V} \rightarrow V_\varepsilon, \quad v \otimes f \mapsto f(1_{(2)} \cdot v)1_{(1)} \rightarrow \hat{1} \quad (2.14c)$$

$$\vec{R}_V: V_\varepsilon \rightarrow \vec{V} \otimes V, \quad \varphi^R \mapsto \sum_i f^i \otimes (\varphi^R \rightarrow 1) \cdot v_i \quad (2.14d)$$

where $\{v_i\}$ is a basis in V and $\{f^i\} \subset \hat{V}$ is its dual basis. More precisely, rigidity intertwiners are the appropriate restrictions of the above maps to the subspaces $\bar{V} \times V \subset \bar{V} \otimes V, \dots$, etc.

Proposition 2.2 *For any object V in $\mathbf{mod} A$ the definitions (2.14a-d) provide intertwiners*

$$\begin{aligned} \bar{R}_V &\in \text{Hom}(V_\varepsilon, V \times \bar{V}), & \bar{R}'_V &\in \text{Hom}(\bar{V} \times V, V_\varepsilon) \\ \vec{R}_V &\in \text{Hom}(V_\varepsilon, \vec{V} \times V), & \vec{R}'_V &\in \text{Hom}(V \times \vec{V}, V_\varepsilon) \end{aligned}$$

satisfying the rigidity equations

$$U_V'^R \circ (\mathbf{1}_V \times \bar{R}'_V) \circ (\bar{R}_V \times \mathbf{1}_V) \circ U_V^L = \mathbf{1}_V \quad (2.15)$$

$$U_{\bar{V}}'^L \circ (\bar{R}'_V \times \mathbf{1}_{\bar{V}}) \circ (\mathbf{1}_{\bar{V}} \times \bar{R}_V) \circ U_{\bar{V}}^R = \mathbf{1}_{\bar{V}} \quad (2.16)$$

$$U_V'^L \circ (\vec{R}'_V \times \mathbf{1}_V) \circ (\mathbf{1}_V \times \vec{R}_V) \circ U_V^R = \mathbf{1}_V \quad (2.17)$$

$$U_{\vec{V}}'^R \circ (\mathbf{1}_{\vec{V}} \times \vec{R}'_V) \circ (\vec{R}_V \times \mathbf{1}_{\vec{V}}) \circ U_{\vec{V}}^L = \mathbf{1}_{\vec{V}}. \quad (2.18)$$

Proof: : The calculation proving left rigidity is this.

\bar{R}_V is an intertwiner:

$$\begin{aligned} \bar{R}_V(x \rightarrow \varphi^R) &= (1 \leftarrow (x \rightarrow \varphi^R)) \cdot v_i \otimes f^i = \langle \varphi^R, 1_{(1)}x \rangle 1_{(2)} \cdot v_i \otimes f^i = \\ &= (x_{(1)} \leftarrow \varphi^R) S(x_{(2)}) \cdot v_i \otimes f^i = (x_{(1)} \leftarrow \varphi^R) \cdot v_i \otimes f^i \cdot S(x_{(2)}) = \\ &= x \cdot \bar{R}_V(\varphi^R) \end{aligned}$$

\bar{R}'_V is an intertwiner:

$$\begin{aligned} \bar{R}'_V(x \cdot (f \otimes v)) &= f(S(x_{(1)})1_{(1)}x_{(2)} \cdot v)1_{(2)} \rightarrow \hat{1} = f(\Pi^R(1_{(1)}x) \cdot v)1_{(2)} \rightarrow \hat{1} = \\ &= f(1_{(1)} \cdot v) \Pi^L(x1_{(2)}) \rightarrow \hat{1} = f(1_{(1)} \cdot v)(x \rightarrow (1_{(2)} \rightarrow \hat{1})) = \\ &= x \cdot \bar{R}'_V(f \otimes v) \end{aligned}$$

The rigidity equation (2.15):

$$\begin{aligned}
LHS &: v \mapsto 1_{(1)} \rightarrow \hat{1} \otimes 1_{(2)} \cdot v \mapsto (1 \leftarrow (1_{(1)} \rightarrow \hat{1})) \cdot v_i \otimes f^i \otimes 1_{(2)} \cdot v \\
&\mapsto S(1_{(1)}) \cdot v_i \otimes f^i(1_{(1')} 1_{(2)} \cdot v) 1_{(2')} \rightarrow \hat{1} \\
&\mapsto ((1_{(2')} \rightarrow \hat{1}) \rightarrow 1) S(1_{(1)}) 1_{(1')} 1_{(2)} \cdot v = \\
&= v
\end{aligned}$$

the rigidity equation (2.16):

$$\begin{aligned}
LHS &: f \mapsto 1_{(1)} \cdot f \otimes 1_{(2)} \rightarrow \hat{1} \mapsto 1_{(1)} \cdot f \otimes (1 \leftarrow (1_{(2)} \rightarrow \hat{1})) \cdot v_i \otimes f^i \\
&\mapsto f(S(1_{(1)}) 1_{(1')} 1_{(2)} \cdot v_i) 1_{(2')} \rightarrow \hat{1} \otimes f^i \\
&\mapsto 1_{(2)} \rightarrow \hat{1} \otimes S^{-1}(1_{(1)}) \cdot f \\
&\mapsto (1 \leftarrow (1_{(2)} \rightarrow \hat{1})) S^{-1}(1_{(1)}) \cdot f = \\
&= f
\end{aligned}$$

The proof of right rigidity is analogous. Q.e.d.

Corollary 2.3 *As a consequence of the rigidity equations we have the left and right duality functors $\mathbf{mod} A \rightarrow \mathbf{mod} A$ mapping $T \in \text{Hom}(V, W)$ into $\bar{T} \in \text{Hom}(\bar{W}, \bar{V})$ and $\vec{T} \in \text{Hom}(\vec{W}, \vec{V})$, respectively, where*

$$\bar{T} := U_{\bar{V}}'^L \circ (\bar{R}'_W \times \mathbf{1}_{\bar{V}}) \circ (\mathbf{1}_{\bar{W}} \times T \times \mathbf{1}_{\bar{V}}) \circ (\mathbf{1}_{\bar{W}} \times \bar{R}_V) \circ U_{\bar{W}}^R \quad (2.19)$$

$$\vec{T} := U_{\vec{V}}^R \circ (\mathbf{1}_{\vec{V}} \times \vec{R}'_W) \circ (\mathbf{1}_{\vec{V}} \times T \times \mathbf{1}_{\vec{W}}) \circ (\vec{R}_V \times \mathbf{1}_{\vec{W}}) \circ U_{\vec{W}}^L \quad (2.20)$$

They are contravariant and antimonoidal and map the K -space $\text{Hom}(V, W)$ isomorphically onto $\text{Hom}(\bar{W}, \bar{V})$ and $\text{Hom}(\vec{W}, \vec{V})$, respectively.

This is a fairly standard result, so the proof is omitted.

It is important to remark that, in spite of the complicated form of the rigidity intertwiners, the left dual \bar{T} of an intertwiner T as well as its right dual \vec{T} , if considered merely as K -linear maps $\hat{W} \rightarrow \hat{V}$, coincide with the transpose of T with respect to the canonical pairing,

$$\langle \bar{T}(f), v \rangle = \langle f, T(v) \rangle = \langle \vec{T}(f), v \rangle, \quad f \in \hat{W}, T \in \text{Hom}(V, W), v \in V. \quad (2.21)$$

This can be checked by explicit calculation using the definitions of \bar{T} , \vec{T} , and those of the intertwiners involved.

Similar phenomenon can be observed if one compares the natural isomorphisms $\vartheta_{V,W}^L: \bar{W} \times \bar{V} \rightarrow \bar{V} \times \bar{W}$, $\vartheta_{V,W}^R: \vec{W} \times \vec{V} \rightarrow \vec{V} \times \vec{W}$ in $\mathbf{mod} A$ with the natural isomorphism $\vartheta_{V,W}: \hat{W} \otimes \hat{V} \rightarrow V \widehat{\otimes} W$ in $\mathbf{vec} K$. As a matter of fact the rigidity intertwiners satisfy the following monoidality relation

$$\bar{R}_{V \times W} = (\mathbf{1}_V \times \mathbf{1}_W \times \vartheta_{V,W}) \circ (\mathbf{1}_V \times \bar{R}_W \times \mathbf{1}_{\bar{V}}) \circ (U_V^R \times \mathbf{1}_{\bar{V}}) \circ \bar{R}_V \quad (2.22a)$$

$$\vec{R}'_{V \times W} = \vec{R}'_W \circ (\mathbf{1}_{\vec{W}} \times U_{\vec{W}}^L) \circ (\mathbf{1}_{\vec{W}} \times \vec{R}'_V \times \mathbf{1}_W) \circ (\vartheta_{V,W}^{-1} \times \mathbf{1}_V \times \mathbf{1}_W) \quad (2.22b)$$

and similar equations for the right rigidity intertwiners. Therefore the forgetful functor $\mathbf{mod} A \rightarrow \mathbf{vec} K$ sends $\vartheta_{V,W}^L$ and $\vartheta_{V,W}^R$ into $\vartheta_{V,W}$.

It is a standard consequence of the existence of left and right duals that there are canonical natural isomorphisms

$$\iota_V := U_{\bar{\bar{V}}}^{\prime L} \circ (\bar{R}'_V \times \mathbf{1}_{\bar{\bar{V}}}) \circ (\mathbf{1}_V \times \bar{R}_{\bar{V}}) \circ U_V^R \in \text{Hom}(V, \bar{\bar{V}}) \quad (2.23a)$$

$$\iota'_V := U_{\bar{\bar{V}}}^{\prime R} \circ (\mathbf{1}_{\bar{\bar{V}}} \times \bar{R}'_V) \circ (\bar{R}_{\bar{V}} \times \mathbf{1}_V) \circ U_V^L \in \text{Hom}(V, \bar{\bar{V}}) \quad (2.23b)$$

Both of these arrows, if considered only as K -linear maps, coincide with the natural isomorphism $V \rightarrow \hat{\hat{V}}$ expressing reflexivity of the objects in $\mathbf{vec} K$, i.e. $\langle \iota_V(v), f \rangle = \langle f, v \rangle = \langle \iota'_V(v), f \rangle$ for all $f \in \hat{\hat{V}}$, $v \in V$.

However, in general one cannot expect to have isomorphic intertwiners $V \rightarrow \bar{\bar{V}}$ in $\mathbf{mod} A$. Equivalently, $\bar{\bar{V}}$ and \bar{V} may not be isomorphic as A -modules. In special WHA's in which the square of the antipode is inner one can still construct natural isomorphisms $\sigma_V^L: V \rightarrow \bar{\bar{V}}$ and $\sigma_V^R: V \rightarrow \bar{\bar{V}}$ but these are not canonical as long as they cannot be expressed in terms of the basic intertwiners $U^{L/R}, \bar{R}, \bar{R}', \dots$, etc. We shall return to this question in case of the C^* -WHA's in Subsection 3.3 where the situation is different due to the existence of a $*$ -operation allowing one to build canonical isomorphisms $\gamma_V: \bar{V} \rightarrow \bar{\bar{V}}$.

A further consequence of the existence of rigidity intertwiners is Frobenius reciprocity. There are two internal Hom's in $\mathbf{mod} A$: $\text{Hom}^L(V, W) := W \times \bar{V}$ represents the functor $Z \mapsto \text{Hom}(Z \times V, W)$ and $\text{Hom}^R(V, W) := \bar{V} \times W$ represents the functor $Z \mapsto \text{Hom}(V \times Z, W)$. Notice that rigidity in the sense of [5], familiar in tensor and quasitensor categories, cannot hold in $\mathbf{rep} A$ since the relation $\text{Hom}^L(X, Y) \times \text{Hom}^L(V, W) \cong \text{Hom}^L(X \times V, Y \times W)$ has no chance in the lack of a braiding.

3 Representations of C^* -WHA's

3.1 $\mathbf{rep} A$ as a bundle over $\mathbf{mod} A$

From now on the number field K is the field \mathbb{C} of complex numbers and the WHA A is assumed to be a C^* -WHA. A *representation* of the C^* -WHA A is a pair $(V, (\cdot, \cdot)_V)$ where V is a finite dimensional vector space over \mathbb{C} carrying a left action of A , i.e. an object of $\mathbf{mod} A$, and $(\cdot, \cdot)_V$ is a scalar product making V a Hilbert space such that the left action of A becomes a $*$ -representation: $(u, x \cdot v)_V = (x^* \cdot u, v)_V$ for all $u, v \in V$ and $x \in A$. The intertwiners from $(V, (\cdot, \cdot)_V)$ to $(W, (\cdot, \cdot)_W)$ are defined to be the intertwiners from V to W in $\mathbf{mod} A$. The category so obtained will be denoted by $\mathbf{rep} A$.

The forgetful functor $\Phi: \mathbf{rep} A \rightarrow \mathbf{mod} A$ sending $(V, (\cdot, \cdot)_V)$ to V is faithful and full and plays the role of a bundle projection. In this and the next subsections we use the shorthand notation V_1, V_2, \dots for objects in the fibre $\Phi^{-1}(\{V\})$. Later the subscripts will be omitted and V also may stand for an object in $\mathbf{rep} A$.

Since any A -module can be made a $*$ -representation by choosing an appropriate scalar product, the fibre over any V of $\mathbf{mod} A$ is non-empty. Since $\mathbf{rep} A$ is a C^* -category, we have a new notion of isomorphism between two representations, the *unitary equivalence*. Consider an isomorphism $T: V \rightarrow W$ in $\mathbf{mod} A$ and choose an object V_1 in the fibre over V . Then there is precisely one object W_1 in the fibre over W such that the lift of T is a unitary equivalence $T_1: V_1 \rightarrow W_1$. We obtain immediately that the fibers, viewed as full subcategories, over isomorphic objects are isomorphic. Furthermore, all objects in

the same fibre are unitarily equivalent. If we fix a V_1 over V while allowing T to run over all automorphisms $V \rightarrow V$ then the polar decomposition $T_1 = H_1 U_1: V_1 \rightarrow V_1$ yields on the one hand all unitaries $U_1: V_1 \rightarrow V_1$ and on the other hand sets up a one-to-one correspondence between the set of objects in the fibre and positive invertible elements H_1 in $\text{End } V_1$.

The monoidal product $V_1 \times W_1$ of V_1 over V and W_1 over W is constructed as follows. One forms the tensor product of Hilbert spaces $V_1 \otimes W_1$ and then defines $V_1 \times W_1$ as the image of the projection $D_{V \times W}(1)$ in $V_1 \otimes W_1$. The monoidal product of intertwiners are defined accordingly. In this way monoidal product becomes a bifunctor preserving the fibres in $\mathbf{rep } A$. As for the unit object in $\mathbf{rep } A$ we have to choose one particular element in the fibre over V_ε . Although all such objects are isomorphic we would like to choose a scalar product which is given by the already existing data in our WHA, namely by the counit.

Lemma 3.1 *The monoidal unit V_ε of $\mathbf{mod } A$, i.e. the left A -module ${}_A \hat{A}^R$, equipped with the scalar product $(\varphi^R, \psi^R) := \hat{\varepsilon}(\varphi^{R*} \psi^R)$ is a $*$ -representation. The maps U_V^L , U_V^R , $U_V'^L$, and $U_V'^R$ of $\mathbf{mod } A$ lift to isometric arrows in $\mathbf{rep } A$ such that $U_{V_1}^L = U_{V_1}^{L*}$ and $U_{V_1}^R = U_{V_1}^{R*}$ for all V_1 in the fibre of V and for all objects V in $\mathbf{mod } A$. These isometric arrows make the unitary representation V_ε a unit object of $\mathbf{rep } A$ (cf. Lemma I.2.12), called the trivial representation.*

Proof: V_ε is a $*$ -representation since

$$\begin{aligned} (\psi_R, x \rightharpoonup \varphi_R) &= \hat{\varepsilon}(\psi_R^*(x \rightharpoonup \varphi_R)) = \hat{\varepsilon}(\hat{S}^{-1}(\psi_R^*)(x \rightharpoonup \varphi_R)) = \\ &= \hat{\varepsilon}(x \rightharpoonup \hat{S}^{-1}(\psi_R^*)\varphi_R) = \hat{\varepsilon}(\hat{S}^{-1}(\psi_R^*)\varphi_R \leftarrow x) = \\ &= \hat{\varepsilon}((\hat{S}^{-1}(\psi_R^*) \leftarrow x)\varphi_R) = \hat{\varepsilon}((S^{-1}(x) \rightharpoonup \psi_R^*)\varphi_R) = \hat{\varepsilon}((x^* \rightharpoonup \psi_R)^*\varphi_R) = \\ &= (x^* \rightharpoonup \psi_R, \varphi_R) . \end{aligned}$$

If V_1 is any $*$ -representation of A and $u, v \in V$ then

$$\begin{aligned} (\varphi_R \otimes u, U_{V_1}^L v) &= \hat{\varepsilon}(\varphi_R^*(1_{(1)} \rightharpoonup \hat{1})) (u, 1_{(2)} \cdot v) = \hat{\varepsilon}((1_{(1)}^* \rightharpoonup \varphi_R)^*) (1_{(2)}^* \cdot u, v) = \\ &= \overline{\langle \varphi_R, 1_{(1)} \rangle} (1_{(2)} \cdot u, v) = ((1 \leftarrow \varphi_R) \cdot u, v) = \\ &= (U_{V_1}^L(\varphi_R \otimes u), v) \end{aligned}$$

hence $U_{V_1}^{L*} = U_{V_1}^L$. Since U_V^L is a bijection with $U_V'^L \circ U_V^L = \mathbf{1}_V$, its lift $U_{V_1}^L$ is an isometry. Similar argument shows that $U_{V_1}^R$ is an isometry, too. The validity of the triangle equations in $\mathbf{rep } A$ follow immediately from that of $\mathbf{mod } A$. *Q.e.d.*

3.2 Duals in $\mathbf{rep } A$

For V_1 a finite dimensional Hilbert space we denote by \hat{V} its dual linear space and by $V_1 \rightarrow \hat{V}$, $u \mapsto \bar{u}$ the antilinear map defined by $\bar{u}(v) := (u, v)$. Let \bar{V}_1 denote the space \hat{V} equipped with the scalar product $(\bar{u}, \bar{v}) := (v, u)$. In this way the isomorphism $u \mapsto \bar{u}$ becomes an antilinear isometry $V \rightarrow \bar{V}$. If V_1 carries a $*$ -representation of the C^* -WHA A , i.e. V_1 is an object of $\mathbf{rep } A$, then there are two natural left A -module structures \bar{V} and \bar{V} on \hat{V} (see Section 2) but neither of them is a $*$ -representation on \bar{V}_1 . If we insist on having duality functors in $\mathbf{rep } A$ that are obtained by lifting the duality functors

of $\mathbf{mod} A$ then we need to modify the scalar product on \bar{V}_1 and must not change its A -module structure. So let \bar{V}_1 and \vec{V}_1 be the objects in the fibre of \bar{V} , resp. \vec{V} , with scalar products

$$(\bar{u}, \bar{v})_{\bar{V}_1} := (v, \Gamma_{V_1} u), \quad (\bar{u}, \bar{v})_{\vec{V}_1} := (v, \Gamma'_{V_1} u), \quad (3.24)$$

where $\Gamma_{V_1}, \Gamma'_{V_1}$ are positive invertible linear transformations of V_1 implementing S^2 . Lifting the left and right rigidity intertwiners $\bar{R}_V, \bar{R}'_V, \vec{R}_V, \vec{R}'_V$ of (2.14a-d) to $\mathbf{rep} A$ we obtain $\bar{R}_{V_1}: V_\varepsilon \rightarrow V_1 \times \bar{V}_1, \dots$ etc. satisfying rigidity relations of the form (2.15, 2.16, 2.17, 2.18) but now in $\mathbf{rep} A$. The corresponding left and right duality functors $T \mapsto \bar{T}$ and $T \mapsto \vec{T}$, can then be defined by lifting formulae (2.19, 2.20) to $\mathbf{rep} A$. As in $\mathbf{mod} A$ so in $\mathbf{rep} A$, the left and right duals \bar{T} and \vec{T} of an intertwiner $T: V_1 \rightarrow W_1$, if considered merely as maps $\hat{W} \rightarrow \hat{V}$, both coincide with the transposed map \hat{T} given by $\langle \hat{T}\hat{w}, v \rangle = \langle \hat{w}, Tv \rangle$.

In a C^* -category it is natural to require that the duality functors be $*$ -functors, i.e. $\bar{T}^* = (\bar{T})^*$ and $\vec{T}^* = (\vec{T})^*$. This implies strong restrictions on the Γ_{V_1} and Γ'_{V_1} in (3.24). For the left dual, for example, this leads to that $T \circ \Gamma_{V_1} = \Gamma_{W_1} \circ T$ must hold for all $T: V_1 \rightarrow W_1$. This implies two things. On the one hand Γ_{V_1} has to be constant on the fibre, and on the other hand it is natural in V . Similar conclusions hold for the right dual. Finally we conclude that there exist positive invertible elements $g_l, g_r \in A$, both of them implementing S^2 , such that the scalar products on all objects V_1 can be written as

$$(\bar{u}, \bar{v})_{\bar{V}_1} := (v, g_l \cdot u), \quad (\bar{u}, \bar{v})_{\vec{V}_1} := (v, g_r^{-1} \cdot u). \quad (3.25)$$

The elements g_l and g_r will be called the *left metric* and the *right metric*, respectively.

Using the $*$ -operation one has more canonical arrows to build out of the $U^{L/R}$ and \bar{R}, \vec{R} intertwiners than it was possible in $\mathbf{mod} A$. In particular the intertwiners

$$\gamma_V := U_V^{R*} \circ (\mathbf{1}_{\bar{V}} \times \bar{R}_V^*) \circ (\vec{R}_V \times \mathbf{1}_{\vec{V}}) \circ U_V^L \in \text{Hom}(\bar{V}, \vec{V}) \quad (3.26)$$

are the components of a natural isomorphism between the left dual and right dual functors. Therefore the intertwiners

$$\sigma_V^L := \bar{\gamma}_V \circ \iota_V : V \rightarrow \bar{\bar{V}} \quad (3.27)$$

$$\sigma_V^R := \vec{\gamma}_V^* \circ \iota'_V : V \rightarrow \vec{\vec{V}} \quad (3.28)$$

provide canonical natural isomorphisms establishing reflexivity in $\mathbf{rep} A$. More precisely, they make the dual object functors in $\mathbf{rep} A$ reflexive in the sense of \mathcal{C} -linear categories. In case of C^* -categories one requires also that σ_V^L, σ_V^R be isometries.

In the next subsections we study the question how to choose the metrics g_l and g_r in order for

- the natural isomorphisms $\gamma_V, \sigma_V^{L/R}$, and ι_V, ι'_V to be isometries,
- γ_V to satisfy sovereignty in the sense of [23],
- and the rigidity intertwiners (2.14a-d) to be standard in the sense of [11].

We shall see that the above unitarity, sovereignty and standardness conditions can be satisfied by unique g_l and g_r and lead to a distinguished choice of the left dual and right dual objects in $\mathbf{rep} A$.

3.3 Sovereignty

A natural equivalence of the fibre preserving $*$ -functors $\overleftarrow{(\quad)}$ and $\overrightarrow{(\quad)}$ is a natural isomorphism $\gamma: \overleftarrow{(\quad)} \rightarrow \overrightarrow{(\quad)}$ in **mod** A with all of its components lifting to isometries $\gamma_{V_1}: \overleftarrow{V_1} \rightarrow \overrightarrow{V_1}$. The intertwiner property $\gamma_V(x \cdot \bar{v}) = x \cdot \gamma_V(\bar{v})$, $x \in A, \bar{v} \in \overleftarrow{V}$, implies that γ_V is the transpose of a map $\gamma_V^t \in \text{End}_{\mathcal{C}} V$ implementing S^2 , i.e. $\gamma_V^t(x \cdot v) = S^2(x) \cdot \gamma_V^t(v)$, $x \in A, v \in V$. Naturality in V , together with semisimplicity of A , leads to that $\gamma_V^t(v) = g' \cdot v$ where $g' \in A$ is independent of V and implements S^2 . Therefore

$$\gamma_V(\bar{v}) = \bar{v} \cdot g' = \overrightarrow{g'^* \cdot v}, \quad v \in V. \quad (3.29)$$

Here we adopted the convention that the antilinear map $v \mapsto \bar{v}$ is denoted by $v \mapsto \bar{v}$ if the image is considered to be the A -module \overleftarrow{V} and by $v \mapsto \bar{v}$ if the image is \overrightarrow{V} . Of course, neither of these maps are A -module maps:

$$x \cdot \bar{v} = \overleftarrow{S(x)^* \cdot v}, \quad x \cdot \bar{v} = \overrightarrow{S(x^*) \cdot v}. \quad (3.30)$$

Now the condition for γ_V to lift to an isometry is that

$$\begin{aligned} (\gamma_V(\bar{u}), \gamma_V(\bar{v}))_{\overleftarrow{V}} &= (\overrightarrow{g'^* \cdot u}, \overrightarrow{g'^* \cdot v})_{\overleftarrow{V}} = \\ &= (g'^* \cdot v, g_r^{-1} g'^* \cdot u)_V = (v, g' g_r^{-1} g'^* \cdot u)_V \end{aligned}$$

be equal to $(\bar{u}, \bar{v})_{\overleftarrow{V}} = (v, g_l \cdot u)_V$ for all $u, v \in V$, i.e.

$$g'^* g' = g_l g_r. \quad (3.31)$$

By definition a natural isomorphism $\gamma: \overleftarrow{(\quad)} \rightarrow \overrightarrow{(\quad)}$ is *monoidal* if

$$\vartheta_{V,W}^R \circ \gamma_W \times \gamma_V = \gamma_{V \times W} \circ \vartheta_{V,W}^L \quad (3.32)$$

and *sovereign* [23] if it is monoidal and satisfies¹

$$\gamma_{\overleftarrow{V}}^{-1} \circ \iota'_V = \bar{\gamma}_V \circ \iota_V. \quad (3.33)$$

Here $\iota'_V: V \rightarrow \overleftarrow{V}$ and $\iota_V: V \rightarrow \overrightarrow{V}$ are the lifts to **rep** A of the natural isomorphisms introduced in (2.23a-b). Using (3.29) the monoidality condition (3.32) and the sovereignty condition (3.33) translate respectively to the following conditions on g' :

$$\Delta(g') = (g' \otimes g') \Delta(1), \quad S(g') = g'^{-1}. \quad (3.34)$$

Such grouplike elements exist in any C^* -WHA therefore sovereignty natural isomorphisms exist in **rep** A , i.e. **rep** A is sovereign. It would be tempting to choose for g' the canonical grouplike element g of Proposition I.4.4. From the point of view of standardness, however, an other choice will be more natural.

Once a choice of the dual objects is made, i.e. g_l and g_r have been fixed, then formula (3.26) yields a canonical choice for γ_V . From now on γ_V will always denote this natural isomorphism.

¹We have relaxed Yetter's condition of γ_{V_e} to be an identity arrow.

Of course, γ_V^{*-1} would also be equally good. So we require γ_V to be isometric. In order to see what this requirement means for the g' underlying expression (3.26) we compute its matrix elements

$$\begin{aligned}
(\vec{v}, \gamma_V(\vec{u}))_{\vec{V}} &= ((\mathbf{1}_{\vec{V}} \times \vec{R}_V) \circ U_{\vec{V}}^R(\vec{v}), (\vec{R}_V \times \mathbf{1}_{\vec{V}}) \circ U_{\vec{V}}^L(\vec{u}))_{\vec{V} \times V \times \vec{V}} = \\
&= ((\mathbf{1}_{\vec{V}} \times \vec{R}_V)(1_{(1)} \cdot \vec{v} \otimes 1_{(2)} \rightarrow \hat{1}), (\vec{R}_V \times \mathbf{1}_{\vec{V}})(1_{(1')} \rightarrow \hat{1} \otimes 1_{(2')} \cdot \vec{u}))_{\vec{V} \times V \times \vec{V}} = \\
&= \sum_{i,j} (\vec{v} \cdot S^{-1}(1_{(1)}) \otimes 1_{(2)} \cdot v_i \otimes \vec{v}_i, \vec{v}_j \otimes 1_{(1')} \cdot v_j \otimes \vec{u} \cdot S(1_{(2')}))_{\vec{V} \times V \times \vec{V}} = \\
&= \sum_i (\vec{v} \cdot S^{-1}(1_{(1)}), \vec{v}_i \cdot 1_{(1')})_{\vec{V}} (\vec{v}_i \cdot 1_{(2)}, \vec{u} \cdot S(1_{(2')}))_{\vec{V}} = \\
&= (\overleftarrow{1_{(2)}^* 1_{(1')} S^{-1}(1_{(1)})^* \cdot v}, \overleftarrow{S(1_{(2')})^* \cdot u})_{\vec{V}} = (g_l \cdot u, g_r^{-1} \cdot v)_V = \\
&= (\vec{v}, \overrightarrow{g_l \cdot u})_{\vec{V}}
\end{aligned}$$

This proves that expression (3.26) corresponds to the choice $g' = g_l$. By Eqn (3.31) this is unitary iff also $g_r = g_l$ holds. For this reason from now on we make the choice $g_l = g_r = g'$ where g' is positive, invertible, and implements S^2 . In order for γ to be sovereign we also require g' to be grouplike. Adjoints of rigidity intertwiners take the simple form

$$\vec{R}_V^* = \vec{R}'_V \circ (\mathbf{1}_V \times \gamma_V) \quad (3.35)$$

$$\vec{R}_V^* = \vec{R}'_V \circ (\gamma_V^{-1} \times \mathbf{1}_V). \quad (3.36)$$

Finally we remark that together with unitarity of γ we also have

Scholium 3.2 *The natural isomorphisms ι , ι' , σ^L , and σ^R of $\mathbf{mod} A$ lift to $\mathbf{rep} A$ as follows.*

$$\iota_V: V \rightarrow \vec{\vec{V}} \quad \iota'_V(v) = \overrightarrow{g_r \cdot v} \quad (3.37)$$

$$\iota'_V: V \rightarrow \vec{\vec{V}} \quad \iota_V(v) = \overleftarrow{g_l^{-1} \cdot v} \quad (3.38)$$

$$\sigma_V^L: V \rightarrow \vec{\vec{V}} \quad \sigma_V^L(v) = \overleftarrow{v} \quad (3.39)$$

$$\sigma_V^R: V \rightarrow \vec{\vec{V}} \quad \sigma_V^R(v) = \overrightarrow{v} \quad (3.40)$$

They all are isometries if we set $g_l = g_r = g'$ where g' is chosen as above.

3.4 Soliton sectors

For a while we postpone the discussion of how to fix the metric g' and turn to the groupoidlike sector composition one meets in WHA's with reducible trivial representation.

Let $\sum_\nu P_\nu$ be the decomposition of the identity arrow of the unit object V_ε into minimal projections in $\text{End } V_\varepsilon$. Then by Proposition I.2.15 it is a sum over vacua and $P_\nu = D_\varepsilon(z_\nu^L)$, $\nu \in \mathcal{V}ac A$.

Lemma 3.3 *If V is an irreducible object of $\mathbf{rep} A$ then there exists one and only one vacuum $\nu \in \mathcal{V}ac A$ such that $P_\nu \times \mathbf{1}_V \neq 0$. This ν depends only on the equivalence class*

q to which V belongs therefore we write $\nu = q^L$ and call it the left vacuum of the sector q . Similarly, there exists one and only one ν , depending only on the class of V , such that $\mathbf{1}_V \times P_\nu \neq 0$. This $\nu = q^R$ is called the right vacuum of q .

Proof: The proof for the left vacuum goes as follows. Let V be an object in $\mathbf{rep} A$ then

$$\mathbf{1}_V = U_V^{L*} \circ (\mathbf{1}_{V_\varepsilon} \times \mathbf{1}_V) \circ U_V^L = \sum_{\nu \in \mathcal{V}ac A} U_V^{L*} \circ (P_\nu \times \mathbf{1}_V) \circ U_V^L \quad (3.41)$$

The right hand side is a sum of mutually orthonal projections $L_V(\nu) \in \text{End } V$. If V is irreducible then $\text{End } V \cong \mathbb{C}$ and there is a unique ν such that $L_V(\nu) \neq 0$. For arbitrary objects V and W and arbitrary $T: V \rightarrow W$ the naturality of the U intertwiners implies that

$$T \circ L_V(\nu) = L_W(\nu) \circ T, \quad \nu \in \mathcal{V}ac A. \quad (3.42)$$

It follows that ν is independent on the choice of the representant V within its equivalence class. *Q.e.d.*

Let us fix a set $\{V_q\}$ of representants in each class q of irreducibles. The short hand notations $\mathbf{1}_p, U_p^L, \bar{R}_p, \dots$ etc will always refer to such representants V_p . For $p, q \in \mathcal{S}ec A$ we consider the monoidal product $V_p \times V_q$. The identity

$$\mathbf{1}_p \times \mathbf{1}_q = \sum_{\nu \in \mathcal{V}ac A} (U_p^{R*} \times \mathbf{1}_q) \circ (\mathbf{1}_p \times P_\nu \times \mathbf{1}_q) \circ (\mathbf{1}_p \times U_q^L)$$

tells us that $V_p \times V_q$ is not the zero object precisely in case of $p^R = q^L$. In particular $(\bar{q})^L = q^R$ and $(\bar{q})^R = q^L$ for all $q \in \mathcal{S}ec A$. If $T: V_r \rightarrow V_p \times V_q$ is a non-zero intertwiner then

$$T \circ L_r(\nu) = T = L_{V_p \times V_q}(\nu) \circ T = (L_p(\nu) \times \mathbf{1}_q) \circ T \quad (3.43)$$

implies that $r^L = p^L$. Similar arguments lead to that every irreducible r occuring in the product $p \times q$ has $r^R = q^R$. Obviously $q^L = q = q^R$ iff q is a vacuum sector.

The irreducible sectors q for which $q^L \neq q^R$ will be called *soliton sectors* since they mimic the behaviour of solitons in 1 + 1-dimensional quantum field theory as long as they connect different vacua and compose accordingly.

The above characterization of soliton sectors is purely categorical. Therefore this soliton structure occurs in any monoidal category with semisimple identity object. For the representation category of a C^* -WHA there is a simple algebraic characterization. The vacua μ are in one-to-one correspondence with minimal projections $z_\mu^L \in Z^L$ and also with minimal projections $z_\mu^R = S(z_\mu^L) \in Z^R$. The left vacuum of the sector $q \in \mathcal{S}ec A$ is the unique μ for which $z_\mu^L e_q = e_q$ and its right vacuum is the unique ν for which $e_q z_\nu^R = e_q$. Using faithfulness of $\varepsilon|_{A^L}$ one can easily verify that

$$\square^L(e_q) = \delta_{q \in \mathcal{V}ac A} \sum_{p \in \mathcal{S}ec A, p^L = q} e_p = z_q^L \delta_{q \in \mathcal{V}ac A} \quad (3.44a)$$

$$\square^R(e_q) = \delta_{q \in \mathcal{V}ac A} \sum_{p \in \mathcal{S}ec A, p^R = q} e_p = z_q^R \delta_{q \in \mathcal{V}ac A}. \quad (3.44b)$$

Let $z_H, H \in \mathcal{H}yp A$ be the minimal hypercentral projections. $\mathcal{H}yp A$ will be called the set of *hyperselection sectors* of A . If z_H is the hypercentral support of z_ν^L , or, what is the same, of z_ν^R then we write $[\nu] = H$. As we have seen $\mathbf{1}_p \times \mathbf{1}_q \neq 0$ implies $p^R = q^L$. Since

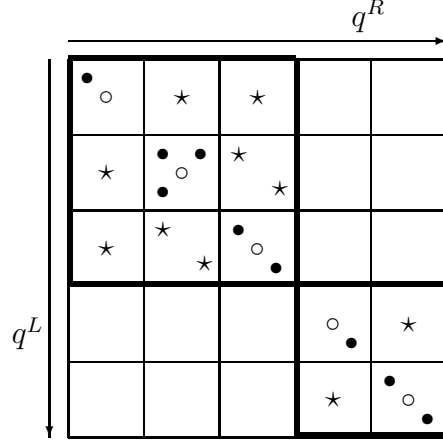


Figure 1: The sector table. The superselection sectors q of A are partitioned into boxes according to their left vacuum (row) and right vacuum (column). Vacuum sectors \circ , soliton sectors \star , and ordinary sectors \bullet . The full submatrices are the hyperselection sectors. Conjugate pairs of sectors are found in transposed positions. The left regular dimension matrix $d_{\mu\nu}$ can be computed as $\sum_{q \in \text{box}} n_q d_q$ with the *box* at the μ -th row and ν -th column.

the left and right vacua of a sector q obviously belong to the same hyperselection sector we obtain that $V_p \times V_q \neq 0$ occurs only for sectors p and q with common hypercentral support.

Now assume $[\mu] = [\nu] = H$ and ask the question whether there exists a sector with $q^L = \mu$ and $q^R = \nu$, i.e. whether $z_\mu^L z_\nu^R \neq 0$. Let \mathcal{V}^H denote the set of vacua in H and let $\mu \sim \nu$ for $\mu, \nu \in \mathcal{V}^H$ be the relation that $z_\mu^L z_\nu^R \neq 0$. Then $\mu \sim \mu$ since $\mu \in \text{Sec } A$ has left and right vacuum just μ . $\mu \sim \nu \Rightarrow \nu \sim \mu$ since q and \bar{q} have left and right vacua interchanged. Finally, $\mu \sim \nu$ and $\nu \sim \lambda$ imply $\mu \sim \lambda$ since $V_p \times V_q \neq 0$ precisely if the middle vacua coincide. Hence \sim is an equivalence relation and one can easily check² that the sum of z_μ^L -s within an equivalence class is S -invariant, hence hypercentral. So it must be the whole of \mathcal{V}^H . This proves that the set of sectors with left vacuum μ and right vacuum ν is non-empty precisely if μ and ν belong to the same hyperselection sector. This will be referred to as the "fullness" of the hypercentral blocks (see Fig. 1).

The hypercentral projections decompose our WHA into a direct sum $\oplus_{H \in \mathcal{H}_{\text{typ}}} A z_H A$ of S -invariant subalgebras that are subcoalgebras as well. Thus the only "interesting" C^* -WHA's are those that have trivial hypercenter. Notice, however, that an "interesting" WHA with more than one vacuum has, in this sense, "non-interesting" sub-WHA's for example the sum of diagonal (i.e. non-solitonic) sectors $A_{\text{diag}} = \oplus_{q, q^L = q^R} e_q A$. Anyhow, throughout the paper we insist on having arbitrary hypercenter.

3.5 Standard rigidity intertwiners

In this subsection the selection of the metric g' will be completed by applying a further principle called *standardness* [11].

Given $g' \in A$ as in the end of Subsection 3.3 the left and right rigidity intertwiners

²For C a class let $z_C := \sum_{\mu \in C} z_\mu^L$. Then $z_C = z_C \sum_{\nu \in \mathcal{V}^H} z_\nu^R = z_C \sum_{\nu \in C} z_\nu^R = z_C S(z_C)$, hence $S(z_C) = z_C$.

(3.35) give rise to the so called left inverse and right inverse

$$\phi_V: \text{End } V \rightarrow \text{End } V_\varepsilon, \quad \phi_V(T) := \overleftarrow{R}'_V \circ (\gamma_V^{-1} \times T) \circ \overrightarrow{R}_V, \quad (3.45)$$

$$\psi_V: \text{End } V \rightarrow \text{End } V_\varepsilon, \quad \psi_V(T) := \overrightarrow{R}'_V \circ (T \times \gamma_V) \circ \overleftarrow{R}_V, \quad (3.46)$$

respectively. These maps are faithful positive traces. This can be seen either by using categorical arguments [11] or by the following direct calculation using the definitions (2.14a-d) and (3.29). At first notice that since $\phi_V(T) = D_\varepsilon(z^L)$ for some $z^L \in Z^L$, $\phi_V(T)\varphi^R = z^L \rightharpoonup \varphi^R = (z^L \rightharpoonup \hat{1})\varphi^R$, so it is sufficient to compute $\phi_V(T)$ on $\hat{1}$.

$$\begin{aligned} \phi_V(T) &: \hat{1} \mapsto \sum_i \overrightarrow{v}_i \otimes v_i \mapsto \sum_i \overleftarrow{g'^{-1}} \cdot v_i \otimes T(v_i) \\ &\mapsto \left\langle \overleftarrow{g'^{-1}} \cdot v_i, 1_{(1)} \cdot T(v_i) \right\rangle 1_{(2)} \rightharpoonup \hat{1} = \left(g'^{-1} \cdot v_i, T(1_{(1)} \cdot v_i) \right)_V 1_{(2)} \rightharpoonup \hat{1} = \\ &= \text{tr}_V(T D_V(g'^{-1} 1_{(1)})) 1_{(2)} \rightharpoonup \hat{1} \\ \phi_V(T) &= D_\varepsilon \left(\text{tr}_V(T D_V(g'^{-1} 1_{(1)})) 1_{(2)} \right) \end{aligned} \quad (3.47)$$

where tr_V denotes trace in the A -module V . Similar expression can be derived for ψ_V . The left inverses depend only on how we fix the freedom in g' . This freedom is multiplying g' with a central positive invertible element c such that $S(c) = c^{-1}$. If A is pure, i.e. V_ε is irreducible, then such a freedom can be eliminated by requiring $\phi_{V_q} = \psi_{V_q}$, $q \in \text{Sec } A$, called the *sphericity* condition in [1]. This corresponds to choosing standard rigidity intertwiners in the sense of [11] in the category $\mathbf{rep } A$. If V_ε is reducible the sphericity condition $\phi = \psi$ cannot hold in general. Our task now is to replace this condition with something that works for reducible identity objects as well.

For V any object in $\mathbf{rep } A$ let $T \in \text{End } V$ and $\nu \in \mathcal{V}ac A$. Then

$$\phi_V(U_V^{R*} \circ (T \times P_\nu) \circ U_V^R) = P_\nu \circ \phi_V(T) \quad (3.48)$$

which can be easily verified by using naturality of U^R and the triangle identities. If V is irreducible, $V \cong V_q$ let us say, then the LHS is zero for $\nu \neq q^R$ therefore $\phi_V(T)$ must be supported on P_{q^R} . This leads to the

Definition 3.4 *Let the left and right dual objects and their rigidity intertwiners be defined by a common choice g' for the left and right metrics. Then g' is called a standard metric if for each $q \in \text{Sec } A$ there is a number d_q such that*

$$\phi_{V_q}(\mathbf{1}_{V_q}) = d_q P_{q^R}, \quad \psi_{V_q}(\mathbf{1}_{V_q}) = d_q P_{q^L}. \quad (3.49)$$

The number d_q is then called the dimension of the sector q .

Although our presentation is a mixture of categorical and weak Hopf algebraic constructions, the above notion of dimension is purely categorical. In fact Eqn (3.49) provides a modification of the notion of standard left-right inverses which works for any monoidal C^* -category with duals in which the selfintertwiner space of the identity object is finite dimensional. It is not our purpose in the present paper to discuss standardness in general but rather to reveal the new phenomena associated to reducibility of the identity object in the representation categories of C^* -WHA's.

So we turn to the determination of the only possible standard metric g' . Surprisingly, this g' does not always coincide with the canonical grouplike element g .

Proposition 3.5 *There exists one and only one standard metric given by the formula*

$$g' = g k_L^{1/2} k_R^{-1/2}, \quad \text{where } k_L = S(k_R) = \sum_{q \in \text{Sec } A} e_q \hat{\varepsilon}(\zeta_{q^L}) = \sum_{\nu \in \text{Vac } A} z_\nu^L \hat{\varepsilon}(\zeta_\nu). \quad (3.50)$$

Proof: Inserting the definitions (2.14a-d) of the the rigidity intertwiners into (3.45) we obtain

$$\phi_V(\mathbf{1}_V) \equiv \vec{R}_V^* \circ \vec{R}_V = D_\varepsilon(\text{tr}_V(g'^{-1} \mathbf{1}_{(1)}) \mathbf{1}_{(2)}) \quad (3.51)$$

$$\psi_V(\mathbf{1}_V) \equiv \vec{R}_V^* \circ \vec{R}_V = D_\varepsilon(\text{tr}_V(g' \mathbf{1}_{(2)}) \mathbf{1}_{(1)}). \quad (3.52)$$

The expressions on the RHS in the argument of the trivial representation D_ε belong to A^L and A^R , respectively. But since D_ε is faithful on these subalgebras and the left hand sides belong to $\text{End } V_\varepsilon$, Proposition I.2.15 imply that these expressions also belong to $\text{Center } A$. If V is set to be the irreducible V_q then this gives, together with the definition of the dimension d_q , that

$$\text{tr}_q(g'^{-1} \mathbf{1}_{(1)}) \mathbf{1}_{(2)} = d_q z_{q^R}^L \quad (3.53)$$

$$\text{tr}_q(g' \mathbf{1}_{(2)}) \mathbf{1}_{(1)} = d_q z_{q^L}^R \quad (3.54)$$

where tr_q stands for tr_{V_q} . Applying the counit to these equations and utilizing the fact that $g' = gc$ with some positive central invertible element $c = \sum_q c_q e_q$, we obtain

$$c_q d_q \varepsilon(z_{q^R}^L) = \text{tr}_q(g^{-1}) = \text{tr}_q(g) = c_q^{-1} d_q \varepsilon(z_{q^L}^R)$$

which determines c_q immediately. But in order to get rid some of the disturbing L, R indices we switch to the dual WHA using the canonical isomorphisms of Z^L and Z^R with \hat{Z} (Lemma I.2.14). According to this Lemma there is a one-to-one correspondence $\nu \mapsto \zeta_\nu$ of vacuum sectors of A and minimal projections of $\hat{Z} = \hat{A}^L \cap \hat{A}^R$. Hence

$$z_\nu^L = 1 \leftarrow \zeta_\nu, \quad z_\nu^R = \zeta_\nu \rightarrow 1, \quad \nu \in \text{Vac } A. \quad (3.55)$$

This proves formula (3.50). *Q.e.d.*

Notice that by the remark after Definition I.4.11 the standard metric g' is also group-like.

Corollary 3.6 *The dimensions of the irreducible objects V_q of $\mathbf{rep } A$ can be expressed in terms of the weak Hopf algebraic data in the following equivalent ways.*

$$d_q = \frac{\text{tr}_q(g')}{\hat{\varepsilon}(\zeta_{q^L})} = \frac{\text{tr}_q(g'^{-1})}{\hat{\varepsilon}(\zeta_{q^R})} = \tau_q / \sqrt{\hat{\varepsilon}(\zeta_{q^L}) \hat{\varepsilon}(\zeta_{q^R})} \quad (3.56)$$

where $\tau_q = \text{tr}_q(g)$.

In Subsection 3.7 we will study the properties of these dimensions. Now for a little while we return to the notion of standardness and formulate it in terms of rigidity intertwiners with equal left dual and right dual objects that is the common practice in C^* -categories. However, the content of the next subsection is not indispensable for the rest of this paper.

3.6 Two-sided duals

Our rigidity intertwiners \overleftarrow{R} , \overrightarrow{R} were lifted from intertwiners of $\mathbf{mod} A$ where they had been associated to different dual objects \overleftarrow{V} and \overrightarrow{V} , respectively. Although in this approach we use *canonical* rigidity intertwiners built out only of those weak Hopf algebraic data that exist for arbitrary fields K , still it is desirable to compare it with an other approach which is more familiar in C^* -categories. Therefore we introduce an "intermediate dual object" that provides a two-sided dual and find the associated standard rigidity intertwiners.

Definition 3.7 For V an object in $\mathbf{rep} A$ let \overline{V} be the dual Hilbert space of V with scalar product $(\bar{u}, \bar{v}) = (v, u)$ and left A -module structure $x \cdot \bar{v} = \bar{v} \cdot g'^{-1/2} S(x) g'^{1/2}$, where g' is the standard metric. \overline{V} is called the conjugate of V .

We can find isometric intertwiners

$$\gamma_V^L: \overleftarrow{V} \rightarrow \overline{V}, \quad \bar{v} \mapsto \bar{v} \cdot g'^{1/2} \quad (3.57a)$$

$$\gamma_V^R: \overrightarrow{V} \rightarrow \overline{V}, \quad \bar{v} \mapsto \bar{v} \cdot g'^{-1/2} \quad (3.57b)$$

which satisfy $(\gamma_V^R)^{-1} \circ \gamma_V^L = \gamma_V$. Therefore the arrows

$$R_V := (\gamma_V^R \times \mathbf{1}_V) \circ \overrightarrow{R}_V: V_\varepsilon \rightarrow \overline{V} \times V \quad (3.58a)$$

$$\bar{R}_V := (\mathbf{1}_V \times \gamma_V^L) \circ \overleftarrow{R}_V: V_\varepsilon \rightarrow V \times \overline{V} \quad (3.58b)$$

satisfy the rigidity relations

$$U_V^{L*} \circ (\overleftarrow{R}_V^* \times \mathbf{1}_V) \circ (\mathbf{1}_V \times R_V) \circ U_V^R = \mathbf{1}_V \quad (3.59a)$$

$$U_V^{R*} \circ (\mathbf{1}_{\overline{V}} \times \overleftarrow{R}_V^*) \circ (R_V \times \mathbf{1}_{\overline{V}}) \circ U_V^L = \mathbf{1}_{\overline{V}} \quad (3.59b)$$

These relations and their adjoints imply that $(\overline{V}, \overleftarrow{R}_V, R_V^*)$ is a left dual and $(\overline{V}, R_V, \overleftarrow{R}_V^*)$ is a right dual of V . This will be briefly referred to as \overline{V} is a *two-sided dual* of V .

The main advantage of the two-sided dual is that the associated left and right dual functors coincide. This can be seen as follows. Let $T: V \rightarrow W$. Then

$$\begin{aligned} & U_V^{L*} \circ (R_W^* \times \mathbf{1}_{\overline{V}}) \circ (\mathbf{1}_{\overline{W}} \times T \times \mathbf{1}_{\overline{V}}) \circ (\mathbf{1}_{\overline{W}} \times \overleftarrow{R}_V) \circ U_W^R = \\ &= \gamma_V^L \circ \bar{T} \circ (\gamma_W^L)^{-1} = \gamma_V^R \circ \gamma_V \circ \bar{T} \circ (\gamma_V)^{-1} \circ (\gamma_W^R)^{-1} = \gamma_V^R \circ \bar{T} \circ (\gamma_W^R)^{-1} = \\ &= U_V^{R*} \circ (\mathbf{1}_{\overline{V}} \times \overleftarrow{R}_W^*) \circ (\mathbf{1}_{\overline{V}} \times T \times \mathbf{1}_{\overline{W}}) \circ (R_V \times \mathbf{1}_{\overline{W}}) \circ U_W^L \end{aligned}$$

We may use the notation \overline{T} for this (left and right) dual of T and call it the *conjugate*. Then conjugation is a linear $*$ -functor, $\overline{\overline{T}} = (T)^*$. Again, as it happened with \bar{T} and \vec{T} , as a map of vector spaces, \overline{T} coincides with the transpose of T and therefore with \bar{T} and \vec{T} , too. The difference is only in the A -module structure and in the scalar product one puts on the vector spaces \hat{V} and \hat{W} .

The rigidity intertwiners (3.58a-b) not only provide a two-sided dual but are also standard. As a matter of fact for all $q \in \text{Sec} A$ we find the normalizations

$$R_{V_q}^* \circ R_{V_q} = d_q D_\varepsilon(z_{q^R}^L) \quad \overleftarrow{R}_{V_q}^* \circ \overleftarrow{R}_{V_q} = d_q D_\varepsilon(z_{q^L}^R) \quad (3.60)$$

and for any finite direct sum $V_i \xrightarrow{T_i} V \xrightarrow{T_i^*} V_i$ of irreducibles $\{V_i\}$

$$R_V = \sum_i (\overline{T_i^*} \times T_i) \circ R_{V_i} \quad (3.61)$$

$$\bar{R}_V = \sum_i (T_i \times \overline{T_i^*}) \circ \bar{R}_{V_i} . \quad (3.62)$$

If $\iota_V: V \rightarrow \hat{V}$ denotes the canonical isomorphism of finite dimensional vector spaces then it lifts to a natural isometric isomorphism $V \rightarrow \overline{\hat{V}}$ in $\mathbf{rep} A$. One has the identities

$$R_{\overline{V}} = (\iota_V \times \mathbf{1}_V) \circ \bar{R}_V , \quad \bar{R}_{\overline{V}} = (\mathbf{1}_{\overline{V}} \times \iota_V) \circ R_V . \quad (3.63)$$

The left and right inverses (3.45) can be expressed in terms of the two-sided duals as follows.

$$\phi_V(T) = R_V^* \circ (\mathbf{1}_{\overline{V}} \times T) \circ R_V \quad (3.64)$$

$$\psi_V(T) = \bar{R}_V^* \circ (T \times \mathbf{1}_{\overline{V}}) \circ \bar{R}_V \quad (3.65)$$

Therefore they are *the* standard left and right inverses in the sense of [11].

3.7 The dimension matrix

Definition 3.8 For V an object in $\mathbf{rep} A$ we define its dimension matrix \mathbf{d}_V as follows. Its rows and columns are labelled by the set $\mathcal{V}ac A$ and

$$\mathbf{d}_V^{\mu\nu} := \sum_{\substack{q \in \mathcal{S}ec A \\ q^L = \mu, \ q^R = \nu}} N_V^q d_q \quad (3.66)$$

where N_V^q denotes the multiplicity of V_q in V .

For pure WHAs when $\mathcal{V}ac A$ has only one element this reduces to the well known dimension formula for a reducible object V . The need for introducing a matrix instead of a scalar in the non-pure case can be seen if we ask about multiplicativity of the dimension. Assume that a positive dimension function $V \mapsto d_V \in \mathbb{R}$ existed which is multiplicative, $d_{V \times W} = d_V d_W$, and additive, $d_V = d_U + d_W$ for V a direct sum of U and W . Then take a soliton sector s and consider the monoidal product $V_s \times V_s$ which is the zero object. Hence $d_s^2 = d_0 = 0$, a contradiction since $d_q \geq 1$ for all $q \in \mathcal{S}ec A$.

The dimension matrix can also be viewed as the set of coefficients for the maps $Z^L \rightarrow Z^R$ and $Z^R \rightarrow Z^L$ provided by the left inverse and the right inverse, respectively, as follows.

$$z_\mu^L \mapsto \phi_V(U_V^{L*} \circ (D_\varepsilon(z_\mu^L) \times \mathbf{1}_V) \circ U_V^L) = D_\varepsilon \left(\sum_\nu \mathbf{d}_V^{\mu\nu} z_\nu^R \right) \quad (3.67)$$

$$z_\mu^R \mapsto \psi_V(U_V^{R*} \circ (\mathbf{1}_V \times D_\varepsilon(z_\mu^R)) \circ U_V^R) = D_\varepsilon \left(\sum_\nu z_\nu^L \mathbf{d}_V^{\nu\mu} \right) \quad (3.68)$$

The very fact that the two sets of coefficients coincide is our standard normalization (3.49).

Proposition 3.9 *The dimension matrix \mathbf{d} is an additive and multiplicative function on the equivalence classes of objects in $\mathbf{rep} A$ on which conjugation acts by transposition. That is to say*

- i) if $V \cong W$ then $\mathbf{d}_V = \mathbf{d}_W$,
- ii) if W is a direct sum of U and V then $\mathbf{d}_W = \mathbf{d}_U + \mathbf{d}_V$,
- iii) $\mathbf{d}_{V \times W} = \mathbf{d}_V \mathbf{d}_W$ for all objects V, W ,
- iv) $\mathbf{d}_{\bar{V}} = \mathbf{d}_V^t$.

Proof: The only non-trivial statement is multiplicativity (iii) which in turn will follow from the next Lemma. *Q.e.d.*

Lemma 3.10 *The (scalar) dimension function d_q given for irreducibles in Corollary 3.6 satisfies the following restricted multiplicativity rule.*

$$d_p \delta_{p^R, q^L} d_q = \sum_{r \in \mathcal{Sec} A} N_{pq}^r d_r, \quad p, q \in \mathcal{Sec} A \quad (3.69)$$

where $N_{pq}^r \equiv N_{V_p \times V_q}^r$ is the multiplicity of r in the product of p and q .

Proof: If $p^R \neq q^L$ then both hand sides are zero since $N_{pq}^r = 0, \forall r \in \mathcal{Sec} A$, in this case. Assume $p^R = q^L$. Calculating $\phi_{V_p \times V_q}(\mathbf{1}_p \times \mathbf{1}_q)$ in two different ways will give the required result. At first using additivity of the left inverse yields the RHS of (3.69). At second use grouplikeness of g' to evaluate (3.47) with $V = V_p \times V_q$. By means of (3.53) we can write

$$\begin{aligned} \phi_{V_p \times V_q}(\mathbf{1}_p \times \mathbf{1}_q) &= \text{tr}_{V_p \times V_q}(g'^{-1} \mathbf{1}_{(1)}) D_\varepsilon(\mathbf{1}_{(2)}) = \\ &= \text{tr}_p(g'^{-1} \mathbf{1}_{(1')}) \text{tr}_q(g'^{-1} \mathbf{1}_{(2')} \mathbf{1}_{(1)}) D_\varepsilon(\mathbf{1}_{(2)}) = \\ &= d_p \text{tr}_q(g'^{-1} z_{p^R}^L \mathbf{1}_{(1)}) D_\varepsilon(\mathbf{1}_{(2)}) = d_p \text{tr}_q(g'^{-1} z_{q^L}^L \mathbf{1}_{(1)}) D_\varepsilon(\mathbf{1}_{(2)}) = \\ &= d_p d_q D_\varepsilon(z_{q^R}^L) \end{aligned}$$

where in the last equality we have utilized the fact that $z_{q^L}^L$ is a projection containing e_q as a subprojection. *Q.e.d.*

3.8 The Frobenius–Schur indicator

In case of a finite group G the Frobenius–Schur indicator is the central element $\iota_G := \sum_{g \in G} g^2$ of the group algebra which takes the values 0 or $\pm 1/n_r$ in irreducible representations. Non-zero values occur precisely for the selfconjugate sectors. For C^* -Hopf algebras one can show that $\iota_H = h_{(1)} h_{(2)}$, where h is the Haar measure, obeys the same properties. For a C^* -WHA A we present here two equivalent definitions for the Frobenius–Schur indicator, one of them is purely categorical, the other one is Hopf algebraic. (cf. [7])

The categorical definition goes as follows. Let V be a selfconjugate irreducible object in $\mathbf{rep} A$. Choose an isomorphism $J: V \rightarrow \bar{V}$. Then $\chi_V := J^{-1} \circ \bar{J} \circ \iota_V: V \rightarrow V$ is a number times $\mathbf{1}_V$ and is independent of the choice of J . In particular it is isometric. It is more tricky to show that it is selfadjoint. Using the expression $\chi_V = U_V^{L*} \circ (\bar{R}_V^* \times \mathbf{1}_V) \circ (\mathbf{1}_V \times J \times J^{-1}) \circ (\mathbf{1}_V \times \bar{R}_V) \circ U_V^R$ one can verify by categorical calculus the identity

$\bar{\chi}_V = J \circ \chi_V^* \circ J^{-1}$. Now use the fact that χ_V , being a unitary selfintertwiner of an irreducible object, must be of the form $u\mathbf{1}_V$ where u is a unit length complex number. Inserting this into our identity we obtain $u\mathbf{1}_{\bar{V}} = \bar{u}\mathbf{1}_{\bar{V}}$, hence $u = \pm 1$. This defines for each selfconjugate sector $q = \bar{q}$ a number $\chi_q = \pm 1$. Extending this definition to $q \neq \bar{q}$ as $\chi_q = 0$ we obtain a natural transformation $\chi_V: V \rightarrow V$.

The Hopf algebraic definition is this. Let $\iota_A := h_{(1)}h_{(2)}$. Then $\iota_A \in \text{Center } A$ has values 0 or $\pm 1/\tau_r$ on irreducibles. Again the non-zero values correspond to selfconjugate sectors. One can show that the \pm sign in ι_A for the sector q coincides with the categorically defined χ_q .

$$h_{(1)}h_{(2)} = \sum_{r \in \text{Sec } A} \frac{\chi_r}{\tau_r} e_r \quad (3.70)$$

where $\tau_r = \text{tr}_r g$. For the proof of this fact and also for clarifying the meaning of the \pm sign the following Scholium is useful.

Scholium 3.11 *In a C^* -WHA A there is a system $\{e_r^{\alpha\beta} \mid r \in \text{Sec } A, \alpha, \beta = 1, \dots, n_r\}$ of matrix units such that the action of the antipode takes the form*

$$g^{-1/2}S(e_r^{\alpha\beta})g^{1/2} = \begin{cases} e_{\bar{r}}^{\beta\alpha} & \text{if } \chi_r = 0 \\ e_r^{\beta\alpha} & \text{if } \chi_r = 1 \\ v_r e_r^{\beta\alpha} v_r^{-1} & \text{if } \chi_r = -1 \end{cases} \quad (3.71)$$

where in the last case n_r must be even, $n_r = 2k_r$, and v_r in the basis $\{e_r^{\alpha\beta}\}$ takes the form $\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$ with I denoting the $k_r \times k_r$ unit matrix.

4 Weyl algebras as Jones extensions

Interpreting a C^* -weak Hopf algebra A as the algebra generated by some set of compact, discrete coordinates and its dual \hat{A} as the algebra generated by the associated canonical momenta we may look for the corresponding Heisenberg or rather Weyl type of commutation relations. The answer is the crossed product $A \rtimes \hat{A}$ (the smash product actually) well known for finite groups and Hopf algebras. The novelty of the weak Hopf setting is the emergence of an amalgamation between coordinates and momenta. We have to identify A^R with \hat{A}^L within $\mathcal{W} = A \rtimes \hat{A}$ via the canonical isomorphism $x^R \mapsto (\hat{1} \leftarrow x^R)$ of Lemma I.2.6.

In the Hopf algebra case the Weyl algebra \mathcal{W} is known to be isomorphic to $\text{End}_{\mathcal{C}} A$ which is clearly the Jones extension of the inclusion $\mathcal{C}1 \subset A$ of scalars in the Hopf algebra A . As a weak Hopf generalization we will show that \mathcal{W} is the Jones extension of $A^L \subset A$. The non-trivial result will be that the Markov trace of this inclusion has trace vector $t_q = f_{q^L} d_q f_{q^R}$ which is in general different from the dimension vector d_q . The appearance of the positive weights f_ν on vacua ν and the existence of a multiplicative extension of the dimension to sectors of A^L and A^R reveals a genuine 2-categorical structure underlying the C^* -WHA A . This structure enables us to prove that each connected component of anyone of the inclusions $A^L \subset A$, $A^R \subset A$, $\hat{A}^L \subset \hat{A}$, has the same Perron-Frobenius eigenvalue, i.e. the same minimal index. For pure C^* -WHA-s this minimal index takes the form $\sum_q n_q d_q$ where n_q is the natural number characterizing the size of the block q while $d_q \geq 1$ is the intrinsic dimension of q derived from the category $\mathbf{rep } A$ in Section

3. This generalizes the index $\sum_q d_q^2$ one obtains for C^* -Hopf algebras [10],[19] in which case $n_q = d_q$ are integers.

4.1 The crossed product C^* -algebra $A \rtimes \hat{A}$

Any WHA A is an A^L - A^R -bimodule in the obvious way. By the canonical isomorphisms κ_A^L and κ_A^R of Lemma (I.2.6) \hat{A} becomes an A^R - A^L -bimodule. One can thus form the bimodule tensor products (or amalgamated tensor products) $A \otimes_{A^R} \hat{A}$ and $\hat{A} \otimes_{A^L} A$.

Definition 4.1 *The Weyl algebra $\mathcal{W} = \mathcal{W}(A)$ of a C^* -WHA A is the crossed product $*$ -algebra $A \rtimes \hat{A}$ with respect to the left Sweedler arrow action of \hat{A} on A . This means that $\mathcal{W} = A \otimes_{A^R} \hat{A}$, as a linear space, and the multiplication and $*$ -operation are defined respectively by*

$$(x \otimes \varphi)(y \otimes \psi) := x(\varphi_{(1)} \rightharpoonup y) \otimes \varphi_{(2)}\psi \quad (4.72)$$

$$(x \otimes \varphi)^* := \varphi_{(1)}^* \rightharpoonup x^* \otimes \varphi_{(2)}^* . \quad (4.73)$$

For showing that the above definition is independent of the choice of the representants $x \otimes \varphi$ one needs only the identities of Scholium I.2.7. It is also easy to see that \mathcal{W} contains A and \hat{A} as unital $*$ -subalgebras. As a matter of fact

$$\begin{aligned} (x \otimes \hat{1})(y \otimes \hat{1}) &= x(\hat{1}_{(1)} \rightharpoonup y) \otimes \hat{1}_{(2)} = xy_{(1)}((\hat{1} \leftarrow y_{(2)}) \rightharpoonup 1) \otimes \hat{1} = \\ &= xy_{(1)} \sqcap^R(y_{(2)}) \otimes \hat{1} = xy \otimes \hat{1} \\ (1 \otimes \varphi)(1 \otimes \psi) &= \varphi_{(1)} \rightharpoonup 1 \otimes \varphi_{(2)}\psi = 1 \otimes (\hat{1} \leftarrow (\varphi_{(1)} \rightharpoonup 1))\varphi_{(2)}\psi = \\ &= 1 \otimes \hat{\sqcap}^L(\varphi_{(1)})\varphi_{(2)}\psi = 1 \otimes \varphi\psi \end{aligned}$$

Identifying $x \in A$ with $x \otimes \hat{1}$ and $\varphi \in \hat{A}$ with $1 \otimes \varphi$ the basic commutation relation of the Weyl algebra reads as

$$\varphi x = x_{(1)}\langle x_{(2)}, \varphi_{(1)} \rangle \varphi_{(2)} . \quad (4.74)$$

The following construction will show that \mathcal{W} possesses a faithful $*$ -representation on a Hilbert space therefore it is actually a C^* -algebra.

The left regular A -module ${}_A A$ is a $*$ -representation if we define the scalar product $(x, y) := \langle \hat{h}, x^* y \rangle$ on A . This Hilbert space is denoted by $L^2(A, \hat{h})$. There is an extension π of this left regular representation to \mathcal{W}

$$\pi(x)y := xy \quad (4.75a)$$

$$\pi(\varphi)y := \varphi \rightharpoonup y . \quad (4.75b)$$

This $*$ -representation is called the *standard representation* of $A \rtimes \hat{A}$ associated to the Haar state. In order to prove that π is faithful assume that $\sum_i x_i \otimes \psi_i \in A \otimes \hat{A}$ represents $\sum_i x_i \psi_i \in \text{Ker } \pi$. Then $\sum_i x_i(\psi_i \rightharpoonup y) = 0$ for all $y \in A$ and in particular

$$\begin{aligned} \sum_i x_i(\psi_i \rightharpoonup y_{(2)})S^{-1}(y_{(1)}) &= 0 \\ \sum_i x_i y_{(2)}S^{-1}(y_{(1)})\langle \psi_i, y_{(3)} \rangle &= 0 \\ \sum_i x_i 1_{(1)}\langle \psi_i, 1_{(2)}y \rangle &= 0 \\ \sum_i x_i 1_{(1)} \otimes \psi_i \leftarrow 1_{(2)} &= 0 \\ \sum_i x_i 1_{(1)} \otimes (\hat{1} \leftarrow 1_{(2)})\psi_i &= 0 \end{aligned}$$

Projecting $A \otimes \hat{A}$ onto $A \otimes_{A^R} \hat{A}$ we obtain $\sum_i x_i \psi_i = 0$. Hence π is faithful.

In a similar fashion one can extend the left regular representation of A to a faithful $*$ -representation π' of the other Weyl algebra $\hat{A} \rtimes A$:

$$\pi'(x)y := xy \quad (4.76a)$$

$$\pi'(\varphi)y := y \leftarrow \hat{S}^{-1}(\varphi) . \quad (4.76b)$$

4.2 The Jones triple $A^L \subset A \subset A \rtimes \hat{A}$

In this subsection we show that in the faithful representation π the Weyl algebra is generated by $\pi(A)$ and by the orthogonal projection $A \rightarrow A^L$. This result is a simple application of the general method [9] in the weak Hopf environment. Although we deviate a little bit from the standard procedure by doing the GNS construction with respect to a non-tracial state, namely \hat{h} , as we have discussed in the Appendix, everything works out as in the tracial case because the modular automorphism of the Haar state leaves the smaller algebra (i.e. A^L or A^R) globally invariant.

Since the Haar element \hat{h} is an idempotent we obtain for $x^L \in A^L$ and $y \in A$ that

$$\begin{aligned} (x^L, y) &= \langle \hat{h}, x^{L*} y \rangle = \langle \hat{h}, \hat{h} \rightarrow x^{L*} y \rangle = \langle \hat{h}, x^{L*} (\hat{h} \rightarrow y) \rangle = \\ &= (x^L, E^L(y)) . \end{aligned}$$

Hence the orthogonal projection onto the subspace A^L is precisely the Haar conditional expectation E^L of (I.4.22). On the other hand $E^L = \pi(\hat{h})$ belongs to $\pi(\mathcal{W})$.

Proposition 4.2 *Let $\tilde{\pi}$ denote the representation of A^{op} on the Hilbert space A by right multiplication, $\tilde{\pi}(x)y := yx$. Then*

$$\pi(\mathcal{W}) = \tilde{\pi}(A^L)' = \pi(A) \pi(\hat{h}) \pi(A) \quad (4.77)$$

Hence \mathcal{W} is the Jones extension of $A^L \subset A$ and the Jones projection $\pi(\hat{h})$ induces the Haar conditional expectation via

$$\pi(\hat{h}) \pi(x) \pi(\hat{h}) = \pi(\hat{h}) \pi(E^L(x)) = \pi(E^L(x)) \pi(\hat{h}) . \quad (4.78)$$

Proof: The identity $\varphi \rightarrow (yx^L) = (\varphi \rightarrow y)x^L$ shows that the Weyl algebra is contained in the commutant $\tilde{\pi}(A^L)'$. On the one hand, the commutant is the Jones extension which is known to be generated by $\pi(A)$ and by the projection e_L projecting onto the subspace A^L .

$$\pi(\mathcal{W}) \subset \tilde{\pi}(A^L)' = \langle \pi(A), e_L \rangle \quad (4.79)$$

On the other hand, we have seen above that $e_L = E^L = \pi(\hat{h})$ therefore

$$\langle \pi(A), e_L \rangle \subset \pi(\mathcal{W}) , \quad (4.80)$$

which proves the main assertion. The implementation formula (4.78) is a plain weak Hopf identity while the fact that $\langle \pi(A), e_L \rangle = \pi(A) e_L \pi(A)$ is a general result [21]. *Q.e.d.*

Analogue results hold for the right Haar conditional expectation $E^R: A \rightarrow A^R$ giving rise to the Jones triple $\hat{A} \rtimes A \supset A \supset A^R$ in which the Jones extension is the other Weyl algebra $\mathcal{W}(\hat{A})$.

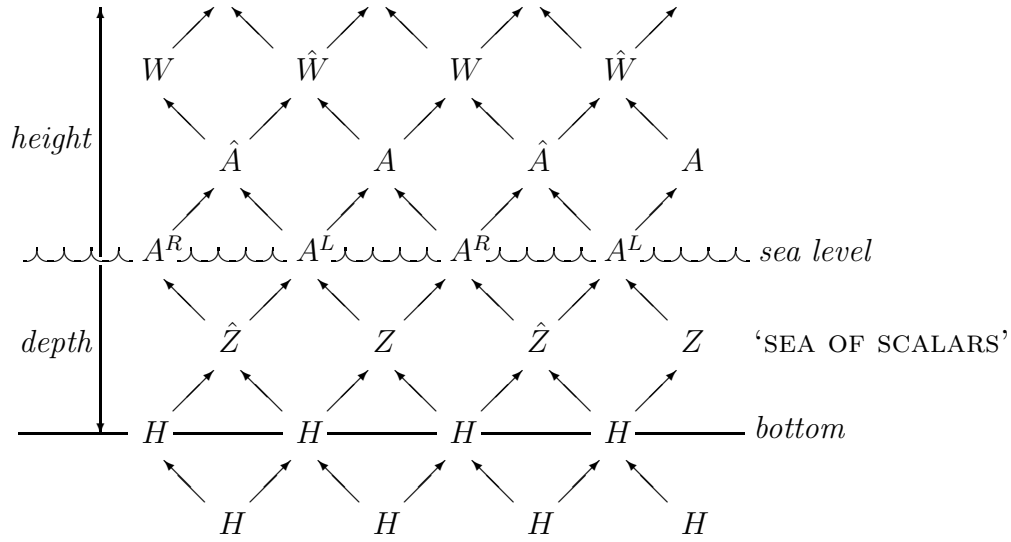


Figure 2: The inclusion diagram of crossed product algebras $A \rtimes \hat{A} \rtimes \dots$ and $\hat{A} \rtimes A \rtimes \hat{A} \dots$ having exactly n terms (A or \hat{A}) at height n above sea level. Each square is a commuting square w.r.t. the Markov conditional expectations E_M^L, E_M^R . Along straight lines from sea level upwards the Jones construction is at work. Starting at depth 2 downwards the bottom is a single copy of the hypercenter.

In order to iterate the basic construction $A^L \subset A \subset \mathcal{W} \subset \dots$ notice that by the left \hat{A}^L -module property of \hat{E}^L we can apply $\text{id}_A \otimes \hat{E}^L$ onto $A \otimes_{A^R} \hat{A}$ to obtain a faithful conditional expectation $\mathcal{W} \rightarrow A$, also denoted by \hat{E}^L . The Jones extension of $A \subset \mathcal{W}$ is the 2-fold iterated crossed product $A \rtimes \hat{A} \rtimes A$ in which the two copies of A commute and the first A with \hat{A} satisfy $\mathcal{W}(A)$ commutation relations while the last A with \hat{A} satisfy $\mathcal{W}(\hat{A})$ commutation relations. Further iterating we obtain a right growing tower of iterated crossed products. Doing the same construction starting with $A \supset A^R$ we obtain a left growing tower but again with the same type of algebras. Putting the two together a 2-dimensional array of inclusions emerges (Figure 2) in which every straight line starting at the "sea level" is a Jones tower and the inclusions as well as the conditional expectations commute around each square. The algebras below the sea level are constructed by taking intersections and their structure is governed by Lemma I.2.14. If A is a Hopf algebra then all algebras at and below the "sea level" coincide with the number field \mathbb{C} . If A is a pure WHA such that \hat{A} is pure as well then only the Z algebras below the sea are equal to \mathbb{C} . But for any WHA A at "depth 2" we reach the bottom which consists of one common copy of Hypercenter A .

Exactly the above array of finite dimensional algebras of "depth 2" arises if one considers the derived towers of a finite index depth 2 inclusion $\mathcal{N} \subset \mathcal{M}$ of von Neumann algebras with relative commutant $\mathcal{N}' \cap \mathcal{M} = A^L$ [15]. The members of the array can also be interpreted as the local algebras associated to intervals in a quantum chain. For Hopf algebras this has been analyzed in [16] and for weak Hopf algebras in [2].

4.3 The Haar conditional expectation

The quasibasis of E^L is by definition an element $\sum_i a_i \otimes b_i \in A \otimes A$ satisfying $\sum_i a_i E^L(b_i x) = x$ for all $x \in A$ [20]. We claim that $\sum_i a_i \otimes b_i = S(h_{(1)}) \otimes g_R^{-2} h_{(2)}$. As a matter of fact

$$\begin{aligned} S(h_{(1)})(\hat{h} \rightarrow (g_R^{-2} h_{(2)} x)) &= S(h_{(1)} S^{-1}(x))(\hat{h} \rightarrow (g_R^{-2} h_{(2)})) = \\ &= x S(h_{(1)}) h_{(2)} \langle \hat{h}, g_R^{-2} h_{(3)} \rangle = x 1_{(1)} \langle \hat{h}, g_L^{-2} h_{(2)} \rangle = \\ &= x 1_{(1)} \varepsilon(g_L^{-2}(\hat{h} \rightarrow h) 1_{(2)}) = x \end{aligned}$$

Hence we obtain for the index of E^L the formula

$$\text{Index } E^L := \sum_i a_i b_i = S(h_{(1)}) g_R^{-2} h_{(2)} = \square^R(g_R^{-2} h) \in A^R \cap \text{Center } A = Z^R \quad (4.81)$$

which is manifestly an element of A^R and belongs to Center A by the general property of the Index [20]. Notice that the connected components of the inclusion $A^L \subset A$ are the inclusions $z_\nu^L A^L \subset z_\nu^L A$ therefore the index being a "scalar" would correspond to $\text{Index } E^L \in Z^L$. By the above formula this is possible only if $\text{Index } E^L$ is hypercentral, i.e. a true scalar in each hyperselection sector. The next Proposition shows that this is indeed the case.

Proposition 4.3 *The index of the Haar conditional expectations $E^L: A \rightarrow A^L$, $x \mapsto (\hat{h} \rightarrow x)$ and $E^R: A \rightarrow A^R$, $x \mapsto (x \leftarrow \hat{h})$ is a common positive invertible hypercentral element I given by*

$$\text{Index } E^L = \sum_{\nu \in \mathcal{V}_{\text{vac}} A} z_\nu^L \frac{\varepsilon(z_\nu^L g_L^{-2})}{\varepsilon(z_\nu^L)} = I = \sum_{\nu \in \mathcal{V}_{\text{vac}} A} z_\nu^R \frac{\varepsilon(z_\nu^R g_R^{-2})}{\varepsilon(z_\nu^R)} = \text{Index } E^R. \quad (4.82)$$

The analogue index in \hat{A} coincides with I under the canonical identification of the hypercenters of A and \hat{A} . If the hypercenter is trivial (especially if A is pure) then the index formula simplifies to $I = 1\varepsilon(g_R^{-2})/\varepsilon(1)$.

Proof: Using Eqns (4.81) and (I.4.13) we obtain

$$I = \sum_q e_q \text{tr}_q(g_L^{-1} g_R^{-1}) / \text{tr}_q(g)$$

Now we need two formulae for ratios of traces of the type $\text{tr}(g^\pm x^{L/R}) / \text{tr}(g)$. For that purpose multiply Eqn (3.53) by $x^R \in A^R$ and Eqn (3.54) by $x^L \in A^L$ and then apply the counit to them. This yields, together with (3.50), the ratios

$$\frac{\text{tr}_q(g^{-1} x^R)}{\text{tr}_q(g^{-1})} = \frac{\langle \zeta_{q^R}, x^R \rangle}{\langle \zeta_{q^R}, 1 \rangle} \quad (4.83)$$

$$\frac{\text{tr}_q(g x^L)}{\text{tr}_q(g)} = \frac{\langle \zeta_{q^L}, x^L \rangle}{\langle \zeta_{q^L}, 1 \rangle}. \quad (4.84)$$

These formulae help to evaluate $\text{tr}(g_L^{-1} g_R^{-1})$ in two different ways which lead immediately to the desired expression for I . *Q.e.d.*

Remark: Since E^L is not a trace preserving conditional expectation (unless \hat{h} is a trace, i.e. $S^2 = \text{id}_A$), scalariness of Index E^L does not necessarily imply any relation between the connected components $z_\nu^L A^L \subset z_\nu^L A$. (See, however, Subsection 3.4.) Thus I may not be the minimal index of $A^L \subset A$.

For later convenience we compute here the Haar conditional expectation E^L on the subalgebra A^R . At first notice that $E^L(x^R) \equiv \hat{h} \rightarrow x^R = 1_{(1)} \langle \hat{h}, 1_{(2)} x^R \rangle$ belongs to $A^L \cap A^R = Z$. So we obtain for the Haar state on the subalgebra $A^L A^R$ the expression

$$\begin{aligned} \langle \hat{h}, x^L y^R \rangle &= \varepsilon(x^L(\hat{h} \rightarrow y^R)) = \sum_{\hat{\nu}} \varepsilon(x^L z_{\hat{\nu}}) \frac{\varepsilon(z_{\hat{\nu}}(\hat{h} \rightarrow y^R))}{\varepsilon(z_{\hat{\nu}})} = \\ &= \sum_{\hat{\nu} \in \mathcal{V}_{ac} \hat{A}} \frac{\varepsilon(x^L z_{\hat{\nu}}) \varepsilon(z_{\hat{\nu}} y^R)}{\varepsilon(z_{\hat{\nu}})} \end{aligned} \quad (4.85)$$

and also

$$\hat{h} \rightarrow x^R = \sum_{\hat{\nu}} z_{\hat{\nu}} \frac{\varepsilon(z_{\hat{\nu}} x^R)}{\varepsilon(z_{\hat{\nu}})}. \quad (4.86)$$

4.4 The Markov trace

Throughout the paper τ denoted the trace on A which is related to the Haar measure by $\tau = g_L^{-1} g_R^{-1} \rightarrow \hat{h}$ and has trace vector $\tau_q = \text{tr}_q(g)$. Since τ is faithful, any other trace is of the form $\tau' = c \rightarrow \tau$ with $c \in \text{Center } A$. If τ' is also faithful then the τ' -preserving conditional expectation $E_{\tau'}^L: A \rightarrow A^L$ can be expressed in terms of the Haar conditional expectation as

$$E_{\tau'}^L(x) = E^L(rx) = E^L(xr), \quad x \in A \quad (4.87)$$

where the Radon-Nikodym derivative r is given by (cf. Eqn (A.2))

$$r = c g_L^{-1} g_R^{-1} E^L(c g_L^{-1} g_R^{-1})^{-1} = c g_R^{-1} (\hat{h} \rightarrow c g_R^{-1})^{-1}. \quad (4.88)$$

The quasibasis and index of $E_{\tau'}^L$ can now be easily obtained,

$$\sum_i a'_i \otimes b'_i = S(h_{(1)}) \otimes r^{-1} g_R^{-2} h_{(2)} \quad (4.89)$$

$$\text{Index } E_{\tau'}^L = S(h_{(1)}) (\tau \rightarrow c) c^{-1} h_{(2)} = \sum_q e_q \frac{\text{tr}_q(g^{-1} c_{(1)}) \langle \tau, c_{(2)} \rangle}{\tau_q c_q} \quad (4.90)$$

where c_q denotes the value of c in the sector q .

An important special case is the *standard trace* τ_S defined by the trace vector d_q . This is obtained by setting in the general trace τ' the central element to be $c = k_L^{-1/2} k_R^{-1/2}$. One would naively expect that the trace with trace vector equal to the dimension vector is nothing else but the Markov trace τ_M , i.e. $\tau_M = \tau_S$ (up to a hypercentral normalization). We will see that this holds only in the absence of soliton sectors.

We recall that the inclusion $A^L \subset A$ is connected iff $Z^L = \mathbb{C}1$, i.e. iff A is pure. Therefore in general there is no unique Markov trace on A but the Markov conditional expectation is unique (see Definition A.3). Let $\Lambda = [\Lambda_{aq}]$, $a \in \text{Sec } A^L$, $q \in \text{Sec } A$ be the inclusion matrix of $A^L \subset A$. Then $\Lambda^t \Lambda$ decomposes into a direct sum of irreducible matrices one for each connected component $z_\nu^L A^L \subset z_\nu^L A$. Hence the row (or column)

indices q of the matrix $\Lambda^t \Lambda$ that belong to one and the same irreducible component can be found in one and the same row in Figure 1. Speaking in terms of Figure 1, on the sectors q of a given row there is a (up to a scalar) unique trace vector which is the Perron-Frobenius eigenvector of the corresponding irreducible component of $\Lambda^t \Lambda$. Any faithful trace with such a trace vector should be called a Markov trace for the inclusion $A^L \subset A$ (or briefly a left Markov trace) since they all share in having the following property. The trace preserving conditional expectation $A \rightarrow A^L$ is independent of the row by row normalization of the trace vector and its index belongs to Z^L . It is a standard result now that the norm of this index is the minimal one among *all* conditional expectations (see Lemma A.2).

If we repeat this construction for the Markov trace of the inclusion $A^R \subset A$ then the resulting trace vector will have the freedom of an overall positive factor in each column of Figure 1. This would be a right Markov trace. Even knowing that the inclusions $A^R \subset A$ and $A^L \subset A$ are isomorphic via the antipode, there seems to be no reason why the left and right Markov traces should coincide. But if they do then they define a trace which is unique up to a scalar in each hyperselection sector. A common left-right Markov trace would also imply a strong relation between the disconnected parts of Λ : They must have the same norm. Therefore that the next Theorem is true comes as an unexpected gift of the weak Hopf structure.

Theorem 4.4 i) *There is a unique trace $\tau_M: A \rightarrow \mathbb{C}$, called the Markov trace such that the τ_M -preserving conditional expectations $E_M^L: A \rightarrow A^L$ and $E_M^R: A \rightarrow A^R$ have equal hypercentral index δ ,*

$$\text{Index } E_M^L = \text{Index } E_M^R = \delta \in \text{Hypercenter } A \quad (4.91)$$

and satisfy the normalization $\tau_M(z_H) = 1$ for $H \in \mathcal{Hyp} A$.

ii) *For any fixed hyperselection sector H the connected inclusions*

$$z_\mu^L A^L \subset z_\mu^L A, \quad A z_\mu^R \supset A^R z_\mu^R, \quad [\mu] = H \quad (4.92)$$

have the same index, i.e. their inclusion matrices $\Lambda(\mu)$ have the same norm. This index is the value δ_H of δ on the hypersector H .

iii) *δ is also equal to the norm of the dimension matrix \mathbf{d}_A of the left regular A -module ${}_A A$. I.e. there exist numbers $f_\mu > 0$, $\mu \in \text{Vac } A$ such that*

$$\sum_{\nu \in \text{Vac } A} d_{\mu\nu} f_\nu = \delta_{[\mu]} f_\mu \quad (4.93)$$

where $d_{\mu\nu}$ stands for $\mathbf{d}_A^{\mu\nu}$ and $[\mu]$ denotes the hypersector of the vacuum μ .

Proof: We have seen in Subsection 3.4 that for μ, ν in the same hypersector H there exists at least one sector q with $q^L = \mu$, $q^R = \nu$. Therefore the H -th block of \mathbf{d}_A is full, i.e. have strictly positive entries. In particular it is an irreducible matrix. Let f_μ , $[\mu] = H$ be a Perron-Frobenius eigenvector and denote its eigenvalue by δ_H .

From the Perron-Frobenius eigenvector we can construct the central element $f_R := \sum_\nu z_\nu^R f_\nu \in Z^R$ and define the trace $\tau_M := c \mapsto \tau$ with $c = c_L c_R$, $c_R = f_R k_R^{-1/2} = S(c_L) \in$

Z^R . Then the general index formula (4.90) yields

$$\begin{aligned}
\text{Index } E_M^L &= \sum_q e_q \frac{\text{tr}_q(g^{-1}1_{(1)})}{\text{tr}_q(g^{-1})} \frac{\langle \tau, c_R 1_{(2)} \rangle}{f_{q^R} \hat{\varepsilon}(\zeta_{q^R})^{-1/2}} = \\
&= \sum_q e_q \frac{\varepsilon(z_{q^R}^L 1_{(1)})}{\varepsilon(z_{q^R}^L)} \frac{\langle \tau, c_R 1_{(2)} \rangle}{f_{q^R} \varepsilon(z_{q^R}^L)^{-1/2}} = \sum_q e_q \frac{\langle \tau, c_R z_{q^R}^L \rangle}{f_{q^R} \varepsilon(z_{q^R}^L)^{1/2}} = \\
&= \sum_\mu z_\mu^R \frac{\langle \tau, c_R z_\mu^L \rangle}{f_\mu \varepsilon(z_\mu^L)^{1/2}} = \sum_\mu z_\mu^R \sum_{q: q^L = \mu} n_q d_q \frac{f_{q^R}}{f_{q^L}} = \\
&= \sum_\mu z_\mu^R \sum_\nu d_{\mu\nu} \frac{f_\nu}{f_\mu}
\end{aligned}$$

which is precisely the hypercentral element δ the components of which are the Perron-Frobenius eigenvalues δ_H .

The restriction of E_M^L onto $z_\mu^L A$ is a trace preserving conditional expectation onto $z_\mu^L A^L$ with scalar index $\delta_{[\mu]}$. Therefore the trace vector is the Perron-Frobenius eigenvector and $\delta_{[\mu]}$ is the corresponding eigenvalue of $\Lambda(\mu)^t \Lambda(\mu)$ (cf. Scholium A.4).

Since τ and c are invariant under the antipode, so is the Markov trace, $\tau_M \circ S = \tau_M$. Therefore

$$E_M^R = S \circ E_M^L \circ S^{-1} \quad (4.94)$$

and their indices are also related by the antipode. Since the index δ is hypercentral, they have equal index. *Q.e.d.*

The Markov trace on A can now be written in the following equivalent forms

$$\tau_M = f_L f_R \rightharpoonup \tau_S = f_L k_L^{-1/2} k_R^{-1/2} f_R \rightharpoonup \tau = f_L k_L^{-1/2} g_L^{-1} g_R^{-1} k_R^{-1/2} f_R \rightharpoonup \hat{h}. \quad (4.95)$$

Hence the trace vector of τ_M is

$$t_q := f_{q^L} d_q f_{q^R}. \quad (4.96)$$

The normalization of τ_M given in the Theorem corresponds to the normalization of f according to

$$\tau_M(z_H) = \sum_{q \in H} n_q t_q = \sum_{\mu, \nu, [\nu] = H} d_{\mu\nu} f_\mu f_\nu = \delta_H \sum_{\mu, [\mu] = H} f_\mu^2 = 1. \quad (4.97)$$

4.5 Dimensions for A^L, A^R

The dimensions $d_q, q \in \text{Sec} A$ have been obtained from the rigid monoidal structure of the category $\text{Rep} A$. Dimensions d_a for the sectors a of A^L cannot be obtained that way since A^L is not a coalgebra hence $\mathbf{rep} A^L$ is not monoidal. However, there is an underlying 2-category \mathcal{C}_A in "dual" position with respect to the representation categories in the sense that A, \hat{A}, A^L, \dots etc are selfintertwiner algebras of certain 1-morphisms (arrows) of \mathcal{C}_A . We will not enter into a precise construction of this 2-category here just give a sketch of its structure on Fig. 3.

\mathcal{C}_A has two 0-morphisms denoted Z and \hat{Z} with the notation referring to their self-intertwiner algebras which is $Z = A^L \cap A^R$ and $\hat{Z} = \hat{A}^L \cap \hat{A}^R$, respectively. There

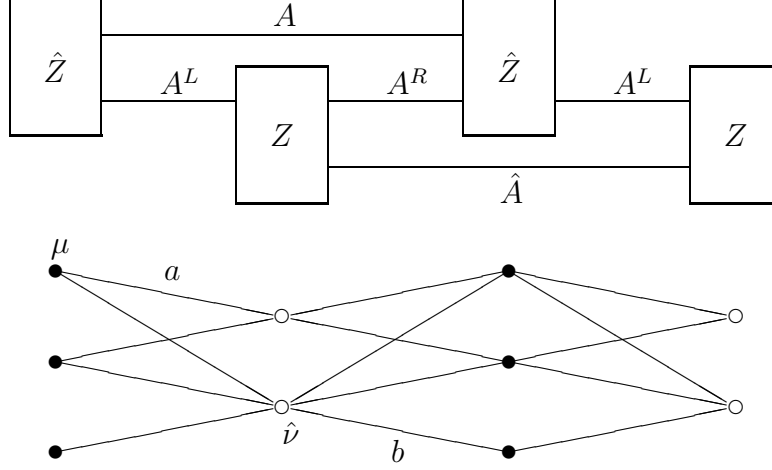


Figure 3: Schematic picture of the 1-skeleton of the 2-category \mathcal{C}_A . Upper half: The reducible 0-objects and 1-objects. Lower half: Their irreducible content.

is a reducible 1-morphism ι pointing from \hat{Z} to Z with algebra A^L and there is one, $\bar{\iota}$, which is its conjugate, pointing from Z to \hat{Z} the associated algebra of which is A^R . Thus the irreducible components $a \in \text{Sec } A^L$ have source $a^L \in \text{Sec } \hat{Z} \equiv \text{Vac } A$ and target $a^R \in \text{Sec } Z \equiv \text{Vac } \hat{A}$. Their conjugates $b = \bar{a} \in \text{Sec } A^R$ have source $b^L \in \text{Vac } \hat{A}$ and target $b^R \in \text{Vac } A$. Figure 3 is an unfolding of this structure in order for the arrows to point to the right and to illustrate the relation with the quantum chain of Figure 2. The WHA A corresponds to the arrow $\iota \times \bar{\iota}$ and its irreducibles to arrows q connecting two vacua of A . Similarly, \hat{A} is the selfintertwiner algebra of $\bar{\iota} \times \iota$ and its irreducibles \hat{q} connect two irreducible components of Z . That there are no more interesting arrows to draw is related to the depth 2 property of Figure 2. The graph with vertex set $\text{Vac } A \cup \text{Vac } \hat{A}$ and edge set $\text{Sec } A^L \cup \text{Sec } A \cup \text{Sec } A^R \cup \text{Sec } \hat{A}$ will be denoted by \mathcal{G}_A .

There are various positive functions defined on the vertices and edges of \mathcal{G}_A . On the vertices we have the function k with values given by the counit evaluated on the minimal projections of Z and \hat{Z} : $k_\mu = \hat{\varepsilon}(\zeta_\mu)$, $k_{\hat{\nu}} = \varepsilon(z_{\hat{\nu}})$. The Perron-Frobenius eigenvector f of the regular dimension matrix (Theorem 4.4) determines the function $\text{Vac } A \ni \mu \mapsto f_\mu$ and the analogue Perron-Frobenius eigenvector $\text{Vac } \hat{A} \ni \hat{\nu} \mapsto f_{\hat{\nu}}$ constructed for \hat{A} extends f to all the vertices of \mathcal{G}_A . These functions determine the weak Hopf algebra elements

$$\begin{aligned}
k &= \sum_{\hat{\nu}} z_{\hat{\nu}} \varepsilon(z_{\hat{\nu}}) \in Z & \hat{k} &= \sum_{\nu} \zeta_{\nu} \hat{\varepsilon}(\zeta_{\nu}) \in \hat{Z} \\
k_L &= 1 \leftarrow \hat{k} \in Z^L & \hat{k}_L &= \hat{1} \leftarrow k \in \hat{Z}^L \\
k_R &= \hat{k} \rightarrow 1 \in Z^R & \hat{k}_R &= k \rightarrow \hat{1} \in \hat{Z}^R \\
f &= \sum_{\hat{\nu}} z_{\hat{\nu}} f_{\hat{\nu}} \in Z & \hat{f} &= \sum_{\nu} \zeta_{\nu} f_{\nu} \in \hat{Z} \\
f_L &= 1 \leftarrow \hat{f} \in Z^L & \hat{f}_L &= \hat{1} \leftarrow f \in \hat{Z}^L \\
f_R &= \hat{f} \rightarrow 1 \in Z^R & \hat{f}_R &= f \rightarrow \hat{1} \in \hat{Z}^R
\end{aligned}$$

On the edges there is the multiplicity function n : n_a determines the dimension of the irrep a of A^L , n_q that of the irrep q of A , For the q and \hat{q} type of edges we already have the dimension function

$$d_q = \frac{\text{tr}_q(g^{-1})}{(k_{q^L} k_{q^R})^{1/2}}, \quad d_{\hat{q}} = \frac{\text{tr}_{\hat{q}}(\hat{g}^{-1})}{(k_{\hat{q}^L} k_{\hat{q}^R})^{1/2}}. \quad (4.98)$$

We now propose an extension of d to the a and b type of edges. But before doing that let us decompose the standard metric into left and right components as

$$g' = g'_L (g'_R)^{-1} \quad g'_L = S(g'_R) = k_L^{1/2} g_L k^{1/2}. \quad (4.99)$$

Analogue formula holds for the dual standard metric \hat{g}' .

Theorem 4.5 i) *There exist unique positive numbers $\{d_a\}_{a \in \text{Sec } A^L}$ $\{d_b\}_{b \in \text{Sec } A^R}$ such that with N_q^{ab} denoting the multiplicity of the simple algebra $e_a^L A^L \otimes e_b^R A^R$ in $e_q A$ we have the multiplicativity rule*

$$\sum_{q \in \text{Sec } A} N_q^{ab} d_q = d_a \delta_{a^R, b^L} d_b \quad (4.100)$$

and the conjugation rule

$$d_{\bar{a}} = d_a. \quad (4.101)$$

Their explicit form is given by

$$d_a = \varepsilon(e_a^L g_L'^{-1})/n_a = \hat{\varepsilon}(\hat{e}_a^R \hat{g}_R'^{-1})/n_a \quad (4.102a)$$

$$d_b = \varepsilon(e_b^R g_R'^{-1})/n_b = \hat{\varepsilon}(\hat{e}_b^L \hat{g}_L'^{-1})/n_b \quad (4.102b)$$

where the minimal central idempotents of A^L , A^R , ... etc are related by $\hat{e}_a^R = e_a^L \rightarrow \hat{1}$, $e_a^L = S(e_{\bar{a}}^R)$, $e_b^R = \hat{e}_b^L \rightarrow 1$. Then it follows that also

$$\sum_{\hat{q} \in \text{Sec } \hat{A}} N_{\hat{q}}^{ba} d_{\hat{q}} = d_b \delta_{b^R, a^L} d_a \quad (4.103)$$

holds with $N_{\hat{q}}^{ba}$ denoting the multiplicity of the simple algebra $\hat{e}_b^L \hat{A}^L \otimes \hat{e}_a^R \hat{A}^R$ in $e_{\hat{q}} \hat{A}$.

ii) Introduce the dimension matrix of A^L , respectively A^R by the formulae

$$d_{\mu\hat{\nu}} := \sum_{\substack{a \in \text{Sec } A^L \\ a^L = \mu, \ a^R = \hat{\nu}}} n_a d_a \equiv \varepsilon(z_\mu^L g_L'^{-1} z_{\hat{\nu}}) \quad (4.104a)$$

$$d_{\hat{\mu}\nu} := \sum_{\substack{b \in \text{Sec } A^R \\ b^L = \hat{\mu}, \ b^R = \nu}} n_b d_b \equiv \varepsilon(z_{\hat{\mu}} g_R'^{-1} z_\nu^R) = d_{\nu\hat{\mu}}. \quad (4.104b)$$

Then the dimension matrices \mathbf{d}_A and $\mathbf{d}_{\hat{A}}$ of the left regular modules of A , respectively \hat{A} can be expressed as

$$d_{\mu\nu} = \sum_{\hat{\rho} \in \text{Vac } \hat{A}} d_{\mu\hat{\rho}} d_{\hat{\rho}\nu}, \quad d_{\hat{\mu}\hat{\nu}} = \sum_{\rho \in \text{Vac } A} d_{\hat{\mu}\rho} d_{\rho\hat{\nu}}. \quad (4.105)$$

iii) The Perron-Frobenius eigenvectors f and \hat{f} are related by

$$\sum_{\mu} d_{\hat{\nu}\mu} f_{\mu} = \delta_{[\hat{\nu}]}^{1/2} f_{\hat{\nu}}. \quad (4.106)$$

Proof: Let us first prove uniqueness of $\{d_a\}$, $\{d_b\}$. Suppose that there exists another solution $\{d'_a\}$, $\{d'_b\}$ of (4.100, 4.101). Then $d'_a/d_a = c_{a^R=b^L} = d_b/d'_b$ for some scalar function $c_{\hat{\nu}}$ on $\text{Vac } \hat{A}$, so $d'_a = c_{a^R} d_a$, $d'_a = d_a/c_{a^R}$. Taking into account the conjugation rule we obtain $c_{\hat{\nu}} \equiv 1$.

In order to verify the solution (4.102a–b) we compute the restriction of the Markov trace onto $A^L A^R$ using (4.85).

$$\begin{aligned} \langle \tau_M, x^L y^R \rangle &= \langle \hat{h}, x^L g_L^{-1} f_L k_L^{-1/2} y^R g_R^{-1} f_R k_R^{-1/2} \rangle = \\ &= \sum_{\hat{\nu}} \varepsilon(x^L f_L k_L^{-1/2} g_L^{-1} z_{\hat{\nu}}) \frac{1}{\varepsilon(z_{\hat{\nu}})} \varepsilon(z_{\hat{\nu}} g_R^{-1} k_R^{-1/2} f_R y^R) = \\ &= \sum_{\hat{\nu}} \varepsilon(x^L f_L g_L'^{-1} z_{\hat{\nu}}) \varepsilon(z_{\hat{\nu}} g_R'^{-1} f_R y^R) . \end{aligned}$$

Specializing to minimal projections we obtain

$$\langle \tau_M, e_a^L e_b^R \rangle = f_{a^L} \varepsilon(e_a^L g_L'^{-1}) \delta_{a^R, b^L} \varepsilon(g_R'^{-1} e_b^R) f_{b^R} . \quad (4.107)$$

This quantity must be equal to $n_a n_b \sum_q N_q^{ab} t_q$. Taking into account the value $t_q = f_{q^L} d_q f_{q^R}$ of the trace vector we arrive to the multiplicativity formula (4.100). The conjugation formula is a simple consequence of the S -invariance of the counit. The second expressions in (4.102a–b), reflecting the symmetric roles of A and \hat{A} , imply (4.103).

(4.105) follows using (4.100), (4.103) and the dimension counting formulae

$$\sum_{a \in \text{Sec } A^L} \sum_{b \in \text{Sec } A^R} N_q^{ab} n_a n_b = n_q \quad \sum_{a \in \text{Sec } A^L} \sum_{b \in \text{Sec } A^R} N_{\hat{q}}^{ba} n_a n_b = n_{\hat{q}} . \quad (4.108)$$

The proof of (4.106) is now straightforward.

Q.e.d.

The importance of the Markov index δ in C^* -WHA's can be illustrated by the three-fold role in which it appears as a Perron-Frobenius eigenvalue:

- δ is the PF-eigenvalue of the left regular dimension matrices \mathbf{d}_A and $\mathbf{d}_{\hat{A}}$ (cf. Theorem 4.4.iii)

$$\sum_{\nu \in \text{Vac } A} d_{\mu\nu} f_{\nu} = \delta_{[\mu]} f_{\mu} \quad \sum_{\hat{\nu} \in \text{Vac } \hat{A}} d_{\hat{\mu}\hat{\nu}} f_{\hat{\nu}} = \delta_{[\hat{\mu}]} f_{\hat{\mu}} .$$

- δ is the PF-eigenvalue of $\Lambda^t \Lambda$ where $\{\Lambda_{aq}\}$ is the inclusion matrix of $A^L \subset A$ (cf. Theorem 4.4.ii),

$$\sum_{q \in \text{Sec } A} \Lambda_{aq} t_q = \delta_{[a]}^{1/2} t_a \quad \sum_{a \in \text{Sec } A^L} t_a \Lambda_{aq} = \delta_{[q]}^{1/2} t_q .$$

Here $t_a = f_{a^L} d_a f_{a^R}$, so $\delta_{[a]}^{1/2} t_a$ is the trace vector of the restriction of τ_M onto A^L .

- δ is also the PF-eigenvalue of the matrix $N_A = [N_{Aq}^r]$ where $N_{Aq}^r = \sum_p n_p N_{pq}^r$ denotes the multiplicity of r in the monoidal product of the left regular module ${}_A A$ with the sector q ,

$$\sum_{r \in \text{Sec } A} N_{Aq}^r t_r = \delta_{[q]} t_q .$$

The identity $S(1_{(1)})1_{(2)} = 1$ occurred many times in this paper but until now nothing has been said about the element $1_{(2)}S(1_{(1)}) \in A^L$. After having introduced g'_L and d_a we are in the position to do this.

Lemma 4.6 *In any WHA A over a field K the element $1_{(2)}S(1_{(1)})$ is the Radon-Nikodym derivative of the left regular trace tr^L of the subalgebra A^L with respect to the nondegenerate functional $\varepsilon|_{A^L}$, i.e.*

$$\text{tr}^L(x^L) = \varepsilon(x^L 1_{(2)}S(1_{(1)})) \quad x^L \in A^L. \quad (4.109)$$

If A is a C^* -WHA then this Radon-Nikodym derivative is positive and invertible and can be expressed as

$$1_{(2)}S(1_{(1)}) = g_L'^{-1} \sum_{a \in \text{Sec } A^L} e_a^L \frac{n_a}{d_a}. \quad (4.110)$$

Proof: Any representation $\Delta(1) = \sum_j e_j \otimes e^j$ with elements $e_j \in A^R$, $e^j \in A^L$ determines a pair of dual bases for the nondegenerate bilinear form $(x^R, x^L) \mapsto \varepsilon(x^R x^L)$ on $A^R \times A^L$ of Lemma I.2.2. Therefore

$$\text{tr}^L(x^L) = \varepsilon(1_{(1)}x^L 1_{(2)}) = \varepsilon(x^L 1_{(2)}S(1_{(1)})) \quad (4.111)$$

due to (I.2.10) and (I.2.2a). In order to prove (4.110) notice that $x^L 1_{(2)}S(1_{(1)}) = 1_{(2)}S(1_{(1)})S^2(x^L)$, $x^L \in A^L$, therefore $w := g'_L 1_{(2)}S(1_{(1)})$ belongs to Center A^L . Then, denoting $\text{tr}_a(x^L) := \frac{1}{n_a} \text{tr}^L(e_a^L x^L)$,

$$d_a \equiv \frac{1}{n_a} \varepsilon(e_a^L g_L'^{-1}) = \frac{1}{n_a} \varepsilon(e_a^L w^{-1} 1_{(2)}S(1_{(1)})) = \text{tr}_a(w^{-1}). \quad (4.112)$$

which proves (4.110).

Q.e.d.

Applying tr_a to (4.110) we obtain

$$d_a = \frac{\text{tr}_a(g_L'^{-1})}{(k_{a^L} k_{a^R})^{1/2}} \quad (4.113)$$

a formula reminiscent to (4.98).

Remark: We summarize without proof some results on the inclusion $A^L A^R \subset A$. Recall that $A^L A^R$ is a C^* -WHA by restricting the structure maps of A and its hypercentral blocks are precisely of the type discussed in Subsection 5.2. The Haar state on $A^L A^R$ is the restriction of \hat{h} and the Haar index is $\text{Index } E^L|_{A^L A^R} = k$, i.e. the element defined in Subsection 4.5. The map

$$E: A \rightarrow A^L A^R, \quad E(x) = k 1_{(1)} E^L(x S^{-1}(1_{(2)})) \quad (4.114)$$

is a conditional expectation which is the \hat{h} -preserving, the τ -preserving, the τ_S -preserving, and the τ_M -preserving conditional expectation at the same time. Its index is hypercentral, thus E is the Markov conditional expectation onto $A^L A^R$.

$$\begin{aligned} \text{Index } E &= \sum_{\mu \in \mathcal{V}_{\text{vac}} A} z_\mu^L \varepsilon(z_\mu^L g_L'^{-2}) \in \text{Hypercenter } A \\ &= \sum_{\mu \in \mathcal{V}_{\text{vac}} A} z_\mu^L \cdot \sum_{a; a^L = \mu} d_a^2 \end{aligned} \quad (4.115)$$

This implies that the above sum of d_a^2 -s is independent of μ within a hyperselection sector H and gives the square of the norm of the inclusion matrix of $z_H A^L A^R \subset z_H A$. (As a comparison, the Haar index I for $A^L \subset A$ can be written as $\sum_{a; a^L=\mu} d_a^2 k_{a^R}$ and the Markov index δ is not algebraically expressible in terms of the dimensions, either.) Since τ_S is a Markov trace for $A^L A^R \subset A$ and has trace vector d_q , the dimension multiplicativity formula (4.100) has as a counterpart

$$\sum_a \sum_b N_q^{ab} d_a d_b = d_q (\text{Index } E)_{[q]} . \quad (4.116)$$

4.6 Temperley-Lieb projections

Since the Weyl algebra $A \rtimes \hat{A}$ is the common Jones extension of the two inclusions $A^L \subset A$ and $\hat{A} \supset \hat{A}^R$, by standard results [9] there exist unique projections \hat{e} and e in $A \rtimes \hat{A}$ that implement the Markov conditional expectations E_M^L and \hat{E}_M^R , respectively, in the sense of the formulae

$$\left. \begin{aligned} \hat{e}x\hat{e} &= E_M^L(x)\hat{e} & x \in A \\ e\varphi e &= \hat{E}_M^R(\varphi)e & \varphi \in \hat{A} \end{aligned} \right\} \text{ within } A \rtimes \hat{A} . \quad (4.117)$$

The peculiarity of the smash product extension is that these Jones projections not only belong to $A \rtimes \hat{A}$ but $e \in A$ and $\hat{e} \in \hat{A}$, as well. Furthermore, they satisfy the Temperley-Lieb relations

$$e\hat{e}e = \delta^{-1}e \quad (4.118a)$$

$$\hat{e}e\hat{e} = \delta^{-1}\hat{e} \quad (4.118b)$$

and, as a consequence of the manifest selfduality of these relations, they also provide us with the Jones projections for the Markov conditional expectations $E_M^R: A \rightarrow A^R$ and $\hat{E}_M^L: \hat{A} \rightarrow \hat{A}^L$ the common Jones extension of which is the other Weyl algebra $\hat{A} \rtimes A$. Therefore

$$\left. \begin{aligned} e\varphi e &= \hat{E}_M^L(\varphi)e \\ \hat{e}x\hat{e} &= E_M^R(x)\hat{e} \end{aligned} \right\} \text{ within } \hat{A} \rtimes A . \quad (4.119)$$

Before proving these statements we recall that in finite dimensional C^* -Hopf algebras it is well-known that $e = h$ and $\hat{e} = \hat{h}$ are the Haar integrals of A and \hat{A} , respectively. Not too much surprisingly this is not true in case of C^* -WHA's. As a matter of fact within $A \rtimes \hat{A}$ we have

$$\hat{h}x\hat{h} = (\hat{h}_{(1)} \rightarrow x)\hat{h}_{(2)}\hat{h} = (\hat{1}_{(1)}\hat{h} \rightarrow x)\hat{1}_{(2)}\hat{h} = (\hat{1}_{(1)} \rightarrow E^L(x))\hat{1}_{(2)}\hat{h} = E^L(x)\hat{h} \quad (4.120)$$

hence $\hat{h}h\hat{h} = g_L^2\hat{h} = \hat{h}g_L^2$. Similarly, in $A \rtimes \hat{A}$ we can write

$$h\varphi h = h\hat{E}^R(\varphi) \quad (4.121)$$

hence $h\hat{h}h = h\hat{g}_R^2 = \hat{g}_R^2h$.

In the next Theorem we use the notions of *standard representation* π_M of $A \rtimes \hat{A}$ and *standard representation* π'_M of $\hat{A} \rtimes A$ associated to the Markov trace. Both of these representations act on the GNS Hilbert space $L^2(A, \tau_M)$ associated to the functional

τ_M . They are equivalent to the standard representations (4.75a-b) and (4.76a-b), respectively, by means of the isometry $U: L^2(A, \tau_M) \rightarrow L^2(A, \hat{h})$, $x \mapsto xs^{1/2}$, where $s = f_L k_L^{-1/2} g_L^{-1} g_R^{-1} k_R^{-1/2} f_R$ is the Radon-Nykodim derivative of τ_M with respect to \hat{h} (4.95). That is to say $\pi_M = \text{Ad}_{U^{-1}} \circ \pi$ and $\pi'_M = \text{Ad}_{U^{-1}} \circ \pi'$.

Theorem 4.7 *The Radon-Nikodym derivatives of the Markov conditional expectations with respect to the Haar ones are given by the following formulae.*

$$E_M^L(x) = E^L(r_R x), \quad x \in A, \quad r_R = \delta^{-1/2} f^{-1} k^{1/2} g_R^{-1} k_R^{-1/2} f_R \quad (4.122a)$$

$$E_M^R(x) = E^R(r_L x), \quad x \in A, \quad r_L = \delta^{-1/2} f_L k_L^{-1/2} g_L^{-1} k^{1/2} f^{-1} \quad (4.122b)$$

In terms of the quantities

$$q_L = \delta^{-1/2} f_L^{-1} k_L^{1/2} g_L^{-1} k^{-1/2} f = 1 \leftarrow \hat{r}_R \quad (4.123a)$$

$$q_R = \delta^{-1/2} f k^{-1/2} g_R^{-1} k_R^{1/2} f_R^{-1} = S(q_L) = \hat{r}_L \rightarrow 1 \quad (4.123b)$$

$$\hat{q}_L = \delta^{-1/2} \hat{f}_L^{-1} \hat{k}_L^{1/2} \hat{g}_L^{-1} \hat{k}^{-1/2} \hat{f} = \hat{1} \leftarrow r_R \quad (4.123c)$$

$$\hat{q}_R = \delta^{-1/2} \hat{f} \hat{k}^{-1/2} \hat{g}_R^{-1} \hat{k}_R^{1/2} \hat{f}_R^{-1} = \hat{S}(\hat{q}_L) = r_L \rightarrow \hat{1} \quad (4.123d)$$

we define the projections

$$e := q_L^{1/2} h q_L^{1/2}, \quad \hat{e} := \hat{q}_L^{1/2} \hat{h} \hat{q}_L^{1/2} \quad (4.124)$$

that are the Jones projections associated to the Markov conditional expectations in the following sense. The standard representations π_M of $A \rtimes \hat{A}$ and π'_M of $\hat{A} \rtimes A$ on the Hilbert space $L^2(A, \tau_M)$ send \hat{e} to the orthogonal projection onto the subspace A^L and A^R , respectively:

$$\pi_M(\hat{e}) = E_M^L, \quad \pi'_M(\hat{e}) = E_M^R. \quad (4.125)$$

e does the same after interchanging the roles of A and \hat{A} . Furthermore e and \hat{e} satisfy the relations (4.117), (4.118a-b), and (4.119).

Proof: Using (A.2) we have $r_R = s E^L(s)^{-1} = f_R k_R^{-1/2} g_R^{-1} (\hat{h} \rightarrow f_R k_R^{-1/2} g_R^{-1})^{-1}$ so we need formula (4.86):

$$\begin{aligned} \hat{h} \rightarrow f_R k_R^{-1/2} g_R^{-1} &= \sum_{\hat{\nu} \in \mathcal{V}_{ac} \hat{A}} z_{\hat{\nu}} \frac{\varepsilon(z_{\hat{\nu}} g_R^{-1} k_R^{-1/2} f_R)}{k_{\hat{\nu}}} = \sum_{\hat{\nu} \in \mathcal{V}_{ac} \hat{A}} \sum_{\mu \in \mathcal{V}_{ac} A} z_{\hat{\nu}} \frac{\varepsilon(z_{\hat{\nu}} g_R^{-1} z_{\mu}^R)}{k_{\hat{\nu}} k_{\mu}^{1/2}} f_{\mu} = \\ &= \sum_{\hat{\nu} \in \mathcal{V}_{ac} \hat{A}} \sum_{\mu \in \mathcal{V}_{ac} A} z_{\hat{\nu}} \frac{d_{\hat{\nu}\mu}}{k_{\hat{\nu}}^{1/2}} f_{\mu} = \delta^{1/2} f k^{-1/2} \end{aligned} \quad (4.126)$$

hence

$$r_R = \delta^{-1/2} f^{-1} k^{1/2} g_R^{-1} k_R^{-1/2} f_R. \quad (4.127)$$

Using (4.94) we obtain $r_L = S(r_R)$.

In order to verify $\pi_M(\hat{e}) = E_M^L$ one uses $\pi_M(\varphi)y = (\varphi \rightarrow y s^{1/2}) s^{-1/2}$, $\varphi \in \hat{A}$, $y \in A$, and the fact that within $A \rtimes \hat{A}$ one has the identification $\hat{q}_L = r_R$. Thus

$$\pi_M(\hat{e})y = r_R^{1/2} E^L(r_R^{1/2} y s^{1/2}) s^{-1/2} = E^L(y r_R^{1/2} s^{1/2}) r_R^{1/2} s^{-1/2} = E^L(y r_R)$$

where we utilized $\theta_{E^L}(r_R) = r_R$ and that $r_R s^{-1} \in A^L$. The proof of $\pi'_M(\hat{e}) = E_M^R$ goes analogously using $\pi'_M(\varphi)y = (ys^{1/2} \leftarrow \hat{S}^{-1}(\varphi))s^{-1/2}$ and the fact that within $\hat{A} \rtimes A$ we have $\hat{q}_R = r_L$. It is now a standard consequence [9] that the relations (4.117) and (4.119) hold true, using also the duality principle for those involving e . The Temperley-Lieb relations (4.118a-b) in turn follow from the fact that $e \in A$ and $\hat{e} \in \hat{A}$ after the reader have checked that $E_M^L(e) = \delta^{-1} = E_M^R(e)$. *Q.e.d.*

Being the Jones extension $A \rtimes \hat{A} = A \hat{e} A$, in particular for every $\varphi \in \hat{A} \subset A \rtimes \hat{A}$ there exist $a_i, b_i \in A$ such that $\varphi = \sum_i a_i \hat{e} b_i$. In order to obtain a concrete expression use the quasibasis $\sum_i u_i \otimes v_i = S(h_{(1)}) \otimes \delta q_R h_{(2)}$ of E_M^L :

$$\begin{aligned} 1 &= \sum_i u_i \hat{e} v_i = u_i \hat{h} r_R v_i \\ \varphi = \varphi 1 &= \sum_i (\varphi_{(1)} \rightarrow u_i) \varphi_{(2)} \hat{h} r_R v_i = \sum_i (\varphi_{(1)} \rightarrow u_i) \hat{\Gamma}^L(\varphi_{(2)}) \hat{h} r_R v_i = \\ &= \sum_i (\varphi \rightarrow u_i) \hat{h} r_R v_i = (\varphi \rightarrow S(h_{(1)})) r_R^{-1/2} \hat{e} r_R^{-1/2} g_R^{-2} h_{(2)} \end{aligned} \quad (4.128)$$

This will be used to prove the following

Corollary 4.8 *Let $\tau_M^W: A \rtimes \hat{A} \rightarrow \mathbb{C}$ be the trace associated to $\tau_M: A \rightarrow \mathbb{C}$ by the basic construction for $A^L \subset A$. Similarly, let $\tau_M^{W'}: A \rtimes \hat{A}$ be the trace associated to the Markov trace t_M of \hat{A} by the basic construction for $\hat{A}^R \subset \hat{A}$. Then $\tau_M^W = \tau_M^{W'}$.*

Notice that, as a consequence of this, the restriction to \hat{A} of the conditional expectation $\hat{E}_M^L: A \rtimes \hat{A} \rightarrow A$ defined by $\hat{E}_M^L(x\hat{e}y) = x\delta^{-1}y$ coincides with the Markov conditional expectation previously denoted by \hat{E}_M^L . Together with the analogue statement for $E_M^R: A \rtimes \hat{A} \rightarrow \hat{A}$, this means commutativity of the Markov conditional expectations around the squares of Figure 2.

Proof: By definition $\tau_M^W(x\hat{e}y) = \delta^{-1}\langle \tau_M, xy \rangle$ for $x, y \in A$ and thus, by cyclicity of the trace, it is sufficient to prove that $\tau_M^W(e\varphi) = \delta^{-1}\langle \varphi, t_M \rangle$ for $\varphi \in \hat{A}$.

$$\begin{aligned} \tau_M^W(e\varphi) &= \sum_i \tau_M^W(e(\varphi \rightarrow u_i) r_R^{-1/2} \hat{e} r_R^{1/2} v_i) = \delta^{-1} \sum_i \tau_M(e(\varphi \rightarrow u_i) v_i) = \\ &= \langle \tau_M, e(\varphi \rightarrow S(h_{(1)})) q_R h_{(2)} \rangle = \langle \tau_M, h q_R^{1/2} (\varphi \rightarrow S(h_{(1)})) q_R h_{(2)} q_R^{1/2} \rangle \end{aligned}$$

where we used the identity $r_R q_R = \delta^{-1} g_R^{-2}$. Inserting here the calculation

$$\tau_M \leftarrow h = h \rightarrow \tau_M = h \rightarrow \hat{h} \hat{g}_L^{-2} \hat{k}^{-1} \hat{f}^2 = \hat{k}^{-1} \hat{f}^2$$

we obtain

$$\begin{aligned} \tau_M^W(e\varphi) &= \varepsilon(q_R^{1/2}(\varphi \rightarrow S(h_{(1)})) q_R h_{(2)} q_R^{1/2} k_R^{-1} f_R^2) = \langle \varphi, S(h_{(1)}) \rangle \varepsilon(q_R^{1/2} \sqcap^R (q_R h_{(2)}) \\ &\quad q_R^{1/2} k_R^{-1} f_R^2) = \langle \varphi, S(h_{(1)}) \rangle \varepsilon(q_R h_{(2)} q_R k_R^{-1} f_R^2) = \langle \varphi, S(q_R h q_R k_R^{-1} f_R^2) \rangle = \\ &= \langle \varphi, f_L^2 k_L^{-1} q_L h q_R \rangle = \delta^{-1} \langle \varphi, t_M \rangle \end{aligned}$$

where in the last equation we took into account the dual of the formula (4.95). *Q.e.d.*

The restrictions of the Markov trace τ_M^W onto various subalgebras of $A \rtimes \hat{A}$ have trace vectors as listed below.

Subalgebra	minimal central projections	trace vector
Z	$z_{\hat{\nu}}, \hat{\nu} \in \mathcal{Vac} \hat{A}$	$\delta_{[\hat{\nu}]} f_{\hat{\nu}}^2$
\hat{Z}	$z_{\mu}, \mu \in \mathcal{Vac} A$	$\delta_{[\mu]} f_{\mu}^2$
A^L	$e_a^L, a \in \mathcal{Sec} A^L$	$\delta_{[a]}^{1/2} f_{a^L} d_a f_{a^R}$
$A^R \equiv \hat{A}^L$	$e_b^R, b \in \mathcal{Sec} A^R$	$\delta_{[b]}^{1/2} f_{b^L} d_b f_{b^R}$
\hat{A}^R	$\hat{e}_a^R, a \in \mathcal{Sec} A^L$	$\delta_{[a]}^{1/2} f_{a^L} d_a f_{a^R}$
A	$e_q, q \in \mathcal{Sec} A$	$f_{q^L} d_q f_{q^R}$
\hat{A}	$\hat{e}_{\hat{q}}, \hat{q} \in \mathcal{Sec} \hat{A}$	$f_{\hat{q}^L} d_{\hat{q}} f_{\hat{q}^R}$
$A \rtimes \hat{A}$	$e_a^W, a \in \mathcal{Sec} A^L$	$\delta_{[a]}^{-1/2} f_{a^L} d_a f_{a^R}$

Some comments on the idempotents e_a^W are in order. From the general theory of Jones extensions we know that $\text{Center } A \rtimes \hat{A} \cong \text{Center } A^L$, however for an unambiguous labelling of the minimal central idempotents of $A \rtimes \hat{A}$ with $a \in \mathcal{Sec} A^L$ we need the "shift isomorphism"

$$\text{Center } A^L \ni z^L \mapsto \sum_i u_i z^L \hat{e} v_i \in \text{Center } A \rtimes \hat{A}. \quad (4.129)$$

Thus our definition is this

$$e_a^W := \sum_i u_i e_a^L \hat{e} v_i = S(h_{(1)}) e_a^L \hat{h} g_R^{-2} h_{(2)}. \quad (4.130)$$

It is important to remark that one would have obtained the same result for e_a^W using the canonical isomorphism $A^L \rightarrow \hat{A}^R$, $e_a^L \mapsto \hat{e}_a^R = e_a^L \rightarrow \hat{1}$ and after that the other shift isomorphism $\text{Center } \hat{A}^R \rightarrow \text{Center } A \rtimes \hat{A}$ associated to the basic construction for $\hat{A}^R \subset \hat{A}$. As a matter of fact the reader may check the following equality in $A \rtimes \hat{A}$ valid for all $z^L \in \text{Center } A^L$,

$$S(h_{(1)}) z^L \hat{h} g_R^{-2} h_{(2)} = \hat{h}_{(1)} \hat{g}_L^{-2} h(z^L \rightarrow \hat{1}) \hat{S}(\hat{h}_{(2)}), \quad (4.131)$$

which expresses commutativity of the triangle consisting of the two shift isomorphisms and of the canonical isomorphism κ_A^L of Lemma I.2.6. Similarly, there is an unambiguous labelling of the minimal central projections $e_b^{W'}$ of $\hat{A} \rtimes A$ by the sectors b of A^R .

The content of the next Proposition can be phrased as Frobenius reciprocity in the underlying 2-category of the weak Hopf algebra.

Proposition 4.9 *For $a, a' \in \mathcal{Sec} A^L$, $b, b' \in \mathcal{Sec} A^R$, and $q \in \mathcal{Sec} A$ let $N_a^{qa'}$ be the multiplicity of the simple algebra $e_q A \otimes \hat{e}_{a'}^R \hat{A}^R$ in the simple algebra $e_a^W (A \rtimes \hat{A})$ and let $N_b^{b'q}$ be the multiplicity of the simple algebra $\hat{e}_b^L \hat{A}^L \otimes e_q A$ in $e_b^{W'} (\hat{A} \rtimes A)$. As before, N_q^{ab} denotes the inclusion matrix of $A^L \otimes_z A^R \subset A$. Then*

$$N_a^{\bar{q}\bar{b}} = N_q^{ab} = N_b^{\bar{a}q} \quad (4.132)$$

where $a \mapsto \bar{a}$ and $b \mapsto \bar{b}$ are the mutually inverse bijections induced by the antipode restricted to $\text{Center } A^{L/R}$, respectively.

Proof: We will content ourselves with proving the first equality. The subalgebra in $A \rtimes \hat{A}$ generated by A and \hat{A}^R is the amalgamated tensor product $A \otimes_{\hat{Z}} \hat{A}^R$ with minimal central projections $e_q \hat{e}_a^R$ where $q^R = a^L$. The inclusion matrix of $A \otimes_{\hat{Z}} \hat{A}^R \subset A \rtimes \hat{A}$ can therefore be computed as follows.

$$\begin{aligned}
N_a^{q\bar{b}} &= \frac{\delta_{[a]}^{1/2}}{t_a} \tau_M^W(e_a^W e_q \hat{e}_b^R) \frac{1}{n_q n_{\bar{b}}} = \frac{\delta_{[a]}^{1/2}}{n_q t_a n_b} \tau_M^W(S(h_{(1)}) e_a^L \hat{e}_b^R r_R^{-1} g_R^{-2} h_{(2)} e_q) = \\
&= \frac{\delta_{[a]}^{-1/2}}{n_q t_a n_b} \tau_M(S(h_{(1)}) e_a^L e_b^R r_R^{-1} g_R^{-2} h_{(2)} e_q) = \\
&= \frac{\delta_{[a]}^{-1/2}}{n_q t_a n_b} \frac{1}{\tau_q} \tau_M(e_q) \text{tr}_q(g^{-1} e_a^L e_b^R r_R^{-1} g_R^{-2}) = \frac{f_{q^L} k_{q^L}^{-1/2} f_{b^L} k_{b^L}^{-1/2}}{t_a n_b} \text{tr}_q(g_L^{-1} e_a^L e_b^R) = \\
&= \frac{k_{a^L}^{-1/2} k_{a^R}^{-1/2}}{d_a n_b} N_q^{ab} \text{tr}_a(g_L^{-1}) \text{tr}_b(1) = N_q^{ab}
\end{aligned}$$

Q.e.d.

Corollary 4.10 *The restriction of an irreducible representation D_q of A onto the subalgebra $z_\mu^L A^L$ is either the zero representation (if $q^L \neq \mu$) or a faithful representation (if $q^L = \mu$). Thus the inclusion matrix Λ of $A^L \subset A$ satisfies*

$$\Lambda_{aq} > 0 \quad \Leftrightarrow \quad a^L = q^L. \quad (4.133)$$

Proof: If $a^L = q^L$ then $\hat{e}_a^L e_q$ is a non-zero projection in $\hat{A} \rtimes A$ (due to the intersection $\hat{A}^L \cap A = \hat{Z}$). Hence there exists a b such that $N_b^{aq} > 0$. It follows from Frobenius reciprocity that $N_q^{ab} > 0$, i.e. $\Lambda_{aq} = \sum_b N_q^{ab} > 0$. *Q.e.d.*

For a pure C^* -WHA this means that every representation represents A^L (and A^R) faithfully. Even in the non-pure case the maximal possible faithfulness is attained which is still compatible with the groupoidlike sector composition.

4.7 Pairing formula

To conclude the general analysis we return to the beginnings and give an expression of the canonical pairing in terms of the Markov trace and of the Temperley-Lieb-Jones projections. This formula can be the starting point of the reconstruction of a WHA from a given inclusion data.

Theorem 4.11 *With τ_M denoting the Markov trace on the Weyl algebra $A \rtimes \hat{A}$ the canonical pairing of $\varphi \in \hat{A}$ and $x \in A$ can be written as*

$$\langle \varphi, x \rangle = \tau_M(\delta^{3/2} \cdot \hat{e} e \varphi g_L''^{1/2} x g_R''^{1/2}) \quad (4.134)$$

where we introduced the notation

$$g_L'' := f_L^{-1} g_L' f^{-1} = f_L^{-1} k_L^{1/2} g_L k_L^{1/2} f^{-1} \quad (4.135a)$$

$$g_R'' := f^{-1} g_R' f_R^{-1} = f^{-1} k_R^{1/2} g_R k_R^{1/2} f_R^{-1} \quad (4.135b)$$

Proof: At first compute $E_M^L(ex) = E^L(r_R q_L^{1/2} h q_L^{1/2} x) = q_L^{1/2} h_{(1)} q_L^{1/2} x_{(1)} \langle \hat{h}, r_R h_{(2)} x_{(2)} \rangle = q_L^{1/2} r_L h_{(1)} S^{-1}(x_{(2)}) q_L^{1/2} x_{(1)} \langle \hat{h}, h_{(2)} \rangle = \delta^{-1} q_L^{-1/2} S^{-1}(x_{(2)}) q_L^{1/2} x_{(1)}$ and then apply this together with (4.128) to obtain

$$\begin{aligned} \hat{E}_M^L(\hat{e}e\varphi) &= \hat{E}_M^L(\hat{e}e(\varphi \rightarrow S(h_{(1)})) r_R^{-1/2} \hat{e} r_R^{-1/2} g_R^{-2} h_{(2)}) = \\ &= \hat{E}_M^L(\hat{e} E_M^L(e(\varphi \rightarrow S(h_{(1)})) r_R^{-1/2}) r_R^{-1/2} g_R^{-2} h_{(2)}) = \\ &= \delta^{-2} q_L^{-1/2} r_L^{-1/2} h_{(2)} q_L^{1/2} S(h_{(3)}) r_R^{-1/2} g_R^{-2} h_{(4)} \langle \varphi, S(h_{(1)}) \rangle = \\ &= \delta^{-2} q_L^{-1/2} r_L^{-1/2} r_R^{-1/2} g_R^{-2} (h \leftarrow \hat{S}(\varphi)) q_L^{1/2} = \\ &= \delta^{-1} g q_R^{1/2} (h \leftarrow \hat{S}(\varphi)) q_L^{1/2}. \end{aligned}$$

Therefore

$$\tau_M(\hat{e}e\varphi x) = \langle \tau_M, \hat{E}_M^L(\hat{e}e\varphi)x \rangle = \langle \tau_M, \delta^{-1} (h \leftarrow \hat{S}(\varphi)) q_L^{1/2} x q_R^{1/2} g \rangle. \quad (4.136)$$

The next step is to express the canonical pairing via the canonical trace τ ,

$$\begin{aligned} \langle \varphi, x \rangle &= \langle \hat{S}(\varphi), (\hat{h} \rightarrow h) g_L^{-2} S^{-1}(x) \rangle = \\ &= \langle \hat{S}(\varphi), h_{(1)} \rangle \langle \hat{h}, h_{(2)} x g_R^{-2} \rangle = \\ &= \langle \tau, (h \leftarrow \hat{S}(\varphi)) x g \rangle. \end{aligned}$$

Taking into account the relation (4.95) of τ_M to τ and then (4.136) we obtain

$$\begin{aligned} \langle \varphi, x \rangle &= \langle \tau_M, (h \leftarrow \hat{S}(\varphi)) x g f_L^{-1} k_L^{1/2} k_R^{1/2} f_R^{-1} \rangle = \\ &= \tau_M(\hat{e}e\varphi \delta f_L^{-1} k_L^{1/2} q_L^{-1/2} x q_R^{-1/2} k_R^{1/2} f_R^{-1}) = \\ &= \tau_M(\delta^{3/2} \hat{e}e\varphi (g_L'')^{1/2} x (g_R'')^{1/2}) \end{aligned}$$

Q.e.d.

5 Special cases

5.1 Weak Kac algebras

Weak Kac algebras (WKA) or, what is the same, generalized Kac algebras of [22] are precisely the weak C^* -Hopf algebras that have involutive antipodes: $S^2 = \text{id}$ [13]. If A is a WKA then its dual weak C^* -Hopf algebra \hat{A} is also a WKA.

Lemma 5.1 *The following conditions for a C^* -WHA A are equivalent.*

- i) A is a WKA, i.e. $S^2 = \text{id}$,
- ii) \hat{h} is a trace on A ,
- iii) $\varepsilon(g_R^{-2}) = \dim A$.

Proof: Equivalence of (i) and (ii) has already been proven in [13]. For completeness we give here an independent argument. At first we recall Subsection I.4.3 that the Haar measure is $\hat{h} = g_L g_R \rightarrow \tau$ where the trace τ has trace vector $\tau_q = \text{tr}_q g \equiv \text{tr}_q g^{-1}$.

(i) \Rightarrow (ii) $S^2 = \text{Ad}_g = \text{id}$ implies $g = 1$ by uniqueness of the canonical grouplike element (Proposition I.4.4). Then $\hat{g} = \hat{1}$ and therefore $\theta_{\hat{h}}(x) = \hat{g} \rightharpoonup x \leftarrow \hat{g} = x$ by (I.4.29), i.e. \hat{h} is a trace.

(ii) \Rightarrow (i) Using the tracial property of \hat{h} and Proposition I.4.9 we have $\hat{h}_{(1)} \otimes \hat{h}_{(2)} = \hat{h}_{(2)} \otimes \hat{h}_{(1)} = \hat{h}_{(1)} \otimes \hat{g} \hat{h}_{(2)} \hat{g}$. By nondegeneracy of \hat{h} this implies $\varphi = \hat{g} \varphi \hat{g}$ for all $\varphi \in \hat{A}$, in particular $\hat{g}^2 = \hat{1}$. Since $\hat{g} \geq 0$, we obtain $\hat{g} = \hat{1}$ and $\hat{S}^2 = \text{id}$. Taking transpose, $S^2 = \text{id}$ follows.

(i) \Leftrightarrow (iii) For $x^L \in A^L$ we have $\varepsilon(x^L) = \langle \hat{h}, x^L \rangle = \tau(x^L g_L g_R)$ implying two interesting identities,

$$\varepsilon(g_R^{-1}) = \tau(g_R) \quad (5.137)$$

$$\varepsilon(g_R^{-2}) = \sum_q \tau_q^2 \quad (5.138)$$

The first one is useful in examples to determine g_R once g is known. The second one together with the inequality

$$\tau_q^2 = (\text{tr}_q g)(\text{tr}_q g^{-1}) \geq n_q^2 \quad (5.139)$$

shows that $\varepsilon(g_R^{-2}) \geq \dim A$ and equality holds iff $\tau_q = n_q$, $q \in \text{Sec } A$, which in turn is equivalent to that $g \in \text{Center } A$ by the well known property of the inequality (5.139). Clearly $g \in \text{Center } A$ iff $S^2 = \text{id}$. *Q.e.d.*

The main result of this subsection is the following

Theorem 5.2 *Let A be a weak Kac algebra. Then the indices of the Markov and of the Haar conditional expectations coincide and take an integer value on each hypercentral block,*

$$I_H = \delta_H \in \mathbb{N}, \quad H \in \mathcal{Hyp} A. \quad (5.140)$$

Moreover, for each vacuum $\mu \in \mathcal{Vac} A$ the dimension of $z_\mu^L A$ is divisible by that of $z_\mu^L A^L$ and their ratio is $I_{[\mu]}$, hence constant over the hypercentral block.

Proof: By Lemma 5.1 ii) the Haar state is tracial. Therefore E^L is a trace preserving conditional expectation the index of which belongs to the common centers of A and A^L . Now Scholium A.4 implies that E^L is the Markov conditional expectation E_M^L , hence $I = \delta$. It remains to show that this common index is

$$\delta_{[\mu]} = \frac{\dim z_\mu^L A}{\dim z_\mu^L A^L} \quad (5.141)$$

and then Lemma A.5 of the Appendix will imply that $I_H = \delta_H$, $H \in \mathcal{Hyp} A$ are integers. For that purpose, and also for mere curiosity, we compute the quantities k_μ , d_q , g_L , f_μ , and $d_{\mu\nu}$ for WKA's. Since $1_{(2)} S(1_{(1)}) = 1$, $\varepsilon|_{A^L} = \text{tr}^L$, the left regular trace of A^L by Lemma 4.6. Furthermore $\tau_q = n_q$ by (5.139). Thus we have

$$k_\nu = \varepsilon(z_\nu^L) = \sum_{a \in \text{Sec } A^L, a^L = \nu} (n_a^L)^2 = \dim(z_\nu^L A^L) \quad (5.142)$$

$$d_q = \frac{\tau_q}{(k_{q^L} k_{q^R})^{1/2}} = \frac{n_q}{(k_{q^L} k_{q^R})^{1/2}}. \quad (5.143)$$

The modular automorphism of the Haar functional on A is the identity therefore $g_L g_R \in \text{Center } A$ by Proposition I.4.14 i). But $g = 1$ implies $g_L = g_R$ therefore $g_L \in A^L \cap A^R \cap \text{Center } A = \text{Hypercenter } A$. Now Proposition 4.3 immediately gives the Haar index

$$I = g_L^{-2} . \quad (5.144)$$

The traces \hat{h} and τ_M having the same trace preserving conditional expectations onto A^L and onto A^R , too, may differ only in a hypercentral Radon-Nikodym derivative. Comparing this to Eqn (4.95) we see that $f_L k_L^{-1/2} k_R^{-1/2} f_R \in \text{Hypercenter } A$ which is possible only if $f_L k_L^{-1/2}$ is itself hypercentral, due to the fullness of the hypercentral blocks. Taking into account the normalization (4.97) this hypercentral element can be determined and yields the expression

$$f_\mu = \left(\frac{k_\mu}{\dim z_{[\mu]} A} \right)^{1/2}, \quad \mu \in \mathcal{V}ac A . \quad (5.145)$$

Therefore the Markov trace has trace vector

$$t_q = \frac{n_q}{\dim z_H A}, \quad q \in H, \quad H \in \mathcal{H}yp A . \quad (5.146)$$

This means that τ_M restricts to the normalized regular trace on each hypercentral block $z_H A$. The regular dimension matrix

$$d_{\mu\nu} = k_\mu^{-1/2} k_\nu^{-1/2} \sum_{\substack{q \in \text{Sec } A \\ q^L = \mu, \quad q^R = \nu}} n_q^2 \quad (5.147)$$

has f as its Perron-Frobenius eigenvector. Inserting 5.145 into the eigenvalue equation one obtains

$$\sum_{q \in \text{Sec } A, \quad q^L = \mu} n_q^2 = k_\mu \cdot \delta_{[\mu]} \quad (5.148)$$

which proves (5.141) and the Theorem. Especially for pure weak Kac algebras we obtain that $\dim A$ is divisible by $\dim A^L$. *Q. e. d.*

5.2 The C^* -WHA $B \otimes B^{op}$

In the Appendix of I. we have shown that any separable algebra B together with a nondegenerate functional E of index 1 determines a WHA structure on $B \otimes B^{op}$. We develop further this construction in case when B is a finite dimensional C^* -algebra and compute the quantities introduced in this paper.

Let $B \cong \oplus_\mu M_{n_\mu}$ with a set of matrix units e_μ^{ij} and minimal central projections e_μ . We define the trace tr on B by setting $\text{tr} e_\mu = n_\mu$ and will also use the traces $\text{tr}_\mu(x) := \text{tr}(e_\mu x)$. The nondegenerate functional $E(x) = \text{tr}(\gamma^2 x)$ is given in terms of a positive invertible $\gamma \in B$ satisfying $\text{tr}_\mu(\gamma^{-2}) = 1$ for all $\mu \in \text{Sec } B$. The structure maps of the C^* -WHA $A = B \otimes B^{op}$ are the following (cf I. Appendix):

$$\Delta(x \otimes y) = \sum_\mu \sum_{ij} (x \otimes e_\mu^{ij} \gamma^{-1}) \otimes (\gamma^{-1} e_\mu^{ji} \otimes y), \quad (5.149)$$

$$\varepsilon(x \otimes y) = \text{tr}(\gamma^2 xy), \quad (5.150)$$

$$S(x \otimes y) = y \otimes \gamma^2 x \gamma^{-2}. \quad (5.151)$$

The left and right subalgebras are $A^L = B \otimes 1$, $A^R = 1 \otimes B$. The sectors of A are pairs (μ, ν) of sectors of B . All (μ, ν) is either a vacuum sector (if $\mu = \nu$) or a soliton sector (if $\mu \neq \nu$). The dual \hat{A} is a simple algebra with a single sector denoted \circ .

Using the Definitions I.3.1 and I.3.24 the reader may check that the element

$$h = \sum_{\mu} \frac{1}{\Gamma_{\mu}} \sum_{ij} e_{\mu}^{ij} \gamma \otimes \gamma e_{\mu}^{ji} \quad (5.152)$$

is the Haar integral in A , where $\Gamma_{\mu} := \text{tr}_{\mu}(\gamma^2)$. Introducing the notation $\Gamma := \sum_{\mu} \Gamma_{\mu} e_{\mu}$ the canonical grouplike element can be written as

$$g = \Gamma^{-1/2} \gamma^2 \otimes \gamma^{-2} \Gamma^{1/2}. \quad (5.153)$$

Hence the trace vector of the canonical trace τ is $\tau_{(\mu, \nu)} = (\text{tr}_{\mu} \otimes \text{tr}_{\nu})(g) = \sqrt{\Gamma_{\mu} \Gamma_{\nu}}$. Using the identity $\varepsilon(g_R^{-1}) = \tau(g_R)$ (5.153) implies

$$g_L = \frac{1}{(\sum_{\mu} \Gamma_{\mu})^{1/2}} \Gamma^{-1/2} \gamma^2 \otimes 1 \quad (5.154a)$$

$$g_R = \frac{1}{(\sum_{\mu} \Gamma_{\mu})^{1/2}} 1 \otimes \gamma^2 \Gamma^{-1/2} \quad (5.154b)$$

In this example $Z^L = \text{Center } A^L$ has minimal idempotents $z_{\mu}^L = e_{\mu} \otimes 1$, therefore the function $k_{\mu} = \Gamma_{\mu}$ and $k_L = \Gamma \otimes 1$, $k_R = 1 \otimes \Gamma$. We obtain for the standard metric the expression $g' = \gamma^2 \otimes \gamma^{-2}$ and the dimensions of all of the sectors are $d_{(\mu, \nu)} = 1$. Taking into account that $k_{\circ} = \varepsilon(1) = \sum_{\nu} \Gamma_{\nu}$ the dimensions of the sectors of A^L and A^R are also trivial: $d_{\mu} = 1$. Hence the left regular dimension matrices of A^L , A , A^R , and \hat{A} are

$$d_{\mu\circ} = n_{\mu}, \quad d_{\mu\nu} = n_{\mu} n_{\nu}, \quad d_{\circ\nu} = n_{\nu}, \quad d_{\circ\circ} = \sum_{\nu} n_{\nu}^2, \quad (5.155)$$

respectively. Hence the Perron-Frobenius eigenvectors are $f_{\mu} = n_{\mu} / \dim B$ and $f_{\circ} = 1 / \sqrt{\dim B}$. The common eigenvalue, which is the Markov index of the inclusions $A^{L/R} \subset A$, is $\delta = \sum_{\nu} n_{\nu}^2 = \dim B = \sqrt{\dim A}$. For generic choices of γ the three traces τ , τ_S , and τ_M are different:

$$\tau(x \otimes y) = \sum_{\mu, \nu} \sqrt{\Gamma_{\mu} \Gamma_{\nu}} (\text{tr}_{\mu} x)(\text{tr}_{\nu} y) \quad (5.156)$$

$$\tau_S(x \otimes y) = (\text{tr } x)(\text{tr } y) \quad (5.157)$$

$$\tau_M(x \otimes y) = \sum_{\mu, \nu} \frac{n_{\mu} n_{\nu}}{(\dim B)^2} (\text{tr}_{\mu} x)(\text{tr}_{\nu} y). \quad (5.158)$$

The Markov trace coincides with the (normalized) trace in the left regular representation of A . The Haar functional \hat{h} and the Haar conditional expectation $E^L(x) = \hat{h} \rightharpoonup x$ are now easy to evaluate,

$$\langle \hat{h}, x \otimes y \rangle = \frac{(\text{tr} \gamma^2 x)(\text{tr} \gamma^2 y)}{\text{tr} \gamma^2}, \quad (5.159)$$

$$E^L(x \otimes y) = x \otimes 1 \frac{\text{tr} \gamma^2 y}{\text{tr} \gamma^2}. \quad (5.160)$$

For the Haar index $I = \text{Index } E^L$ one obtains the scalar $I = 1_A \text{tr} \gamma^2$.

A On the index of finite dimensional inclusions

The following results may belong to the standard part of the theory of inclusions of multimatrix algebras [9], although it is difficult to find them in the form presented here, mainly because we have been using Watatani's ring theoretical notion of index [20].

Scholium A.1 *Let $A \subset B$ be a unital inclusion of finite dimensional C^* -algebras and let $\varphi: B \rightarrow \mathbb{C}$ be a faithful positive linear functional. Define the A -module maps $E: {}_A B \rightarrow {}_A A$ and $F: B_A \rightarrow A_A$ respectively by the formulae*

$$\varphi(aE(b)) = \varphi(ab), \quad \varphi(F(b)a) = \varphi(ba) \quad a \in A, b \in B.$$

Then the following statements are equivalent:

- i) $\theta(A) \subset A$
- ii) $E \circ \theta = \theta \circ F$
- iii) $E = F$
- iv) E is a conditional expectation.

If the above equivalent conditions hold then $E_\varphi = E$ will be called the φ -preserving conditional expectation.

Now let φ and ψ be faithful positive functionals on B such that $\theta_\varphi(A) = A$ and $\theta_\psi(A) = A$. We define the (left) Radon-Nikodym derivatives of φ w.r.t ψ and of E_φ w.r.t E_ψ , respectively by

$$\varphi(b) = \psi(sb), \quad E_\varphi(b) = E_\psi(rb), \quad b \in B. \quad (\text{A.1})$$

r can be computed from s by the following formulae

$$r = E_\varphi(s^{-1})s = E_\psi(s)^{-1}s = sE_\psi(s)^{-1}. \quad (\text{A.2})$$

As a matter of fact $\varphi(E_\varphi(b)a) = \varphi(ba) = \psi(sba) = \psi(E_\psi(sb)a) = \varphi(E_\varphi(s^{-1})E_\psi(sb)a)$, implying the first equality in (A.2). The second follows from the first because $E_\psi(r) = 1$ and the third follows from the second since $r \in A' \cap B$ [20]. One can also show easily that the modular automorphisms are related by

$$\theta_\varphi = \theta_\psi \circ \text{Ad}_s, \quad \theta_{E_\varphi} = \theta_\varphi|_{A' \cap B} = \theta_{E_\psi} \circ \text{Ad}_r|_{A' \cap B}, \quad (\text{A.3})$$

and that r and s commute. If $\psi = \tau$ is tracial then r is positive and we have

$$E_\varphi(b) = E_\tau(rb) = E_\tau(br) = E_\tau(r^{1/2}br^{1/2}), \quad b \in B. \quad (\text{A.4})$$

For a fixed faithful trace τ and for arbitrary φ and ψ as above let s_φ , s_ψ and r_φ , r_ψ be the corresponding Radon-Nikodym derivatives w.r.t τ and E_τ , respectively. Then one has the following manifestly positive expressions

$$\varphi(b) = \psi(s_\psi^{-1/2} s_\varphi^{1/2} b s_\varphi^{1/2} s_\psi^{-1/2}), \quad (\text{A.5})$$

$$E_\varphi(b) = E_\psi(r_\psi^{-1/2} r_\varphi^{1/2} b r_\varphi^{1/2} r_\psi^{-1/2}). \quad (\text{A.6})$$

In order to study the index of various conditional expectations we need the inclusion data $A \subset B$ explicitly: $A \cong \oplus_{\alpha} M_{n_{\alpha}}$, $B \cong \oplus_{\beta} M_{m_{\beta}}$, and inclusion matrix $\Lambda = [\Lambda_{\beta\alpha}]$. Then there exists a set $\{e_{\beta}^{IJ} \mid I, J \in \mathcal{I}_{\beta}, \beta \in \mathcal{Sec} B\}$ of matrix units for B where the index set \mathcal{I}_{β} consists of triples $I = (a, \alpha, i)$ where $\alpha \in \mathcal{Sec} A$, is such that $\Lambda_{\beta\alpha} > 0$, $i = 1, \dots, \Lambda_{\beta\alpha}$, and $a = 1, \dots, n_{\alpha}$. An arbitrary conditional expectation

$$E: B \rightarrow A, \quad E(e_{\beta}^{a'\alpha'i',i\alpha a}) = \delta^{\alpha'\alpha} \Phi_{\beta\alpha}^{i'i} e_{\alpha}^{a'a} \quad (\text{A.7})$$

can be uniquely characterized by positive elements $\Phi_{\beta\alpha} \in M_{\Lambda_{\beta\alpha}}$ satisfying $\sum_{\beta} \text{tr} \Phi_{\beta\alpha} = 1$, $\forall \alpha$. Here

$$e_{\alpha}^{a'a} = \sum_{\beta} \sum_k e_{\beta}^{a'\alpha k, k\alpha a} \quad \alpha \in \mathcal{Sec} A, \quad a', a = 1, \dots, n_{\alpha} \quad (\text{A.8})$$

are matrix units for A . A faithful conditional expectation E corresponds to having $\Phi_{\beta\alpha}$ invertible whenever $\Lambda_{\beta\alpha} > 0$. In the latter case we choose invertible $C_{\beta\alpha}$ such that $\Phi_{\beta\alpha} = C_{\beta\alpha} C_{\beta\alpha}^*$. Then it is straightforward to verify that the set of elements

$$b_{\beta}^{a'\alpha'i',i\alpha a} := \sum_j e_{\beta}^{a'\alpha'i',j\alpha a} \frac{1}{\sqrt{n_{\alpha}}} (C_{\beta\alpha}^{-1*})^{ji} \quad (\text{A.9})$$

form a quasibasis of E , i.e. $\sum_{\beta} \sum_{I, J \in \mathcal{I}_{\beta}} b_{\beta}^{IJ} E(b_{\beta}^{IJ*} x) = x$, $\forall x \in B$. Therefore

$$\text{Index } E = \sum_{I\beta J} b_{\beta}^{IJ} b_{\beta}^{IJ*} = \sum_{\beta\alpha} e_{\beta} \cdot \text{tr} \Phi_{\beta\alpha}^{-1} \quad (\text{A.10})$$

where $e_{\beta} = \sum_I e_{\beta}^{II}$. Let $\{f_{\beta\alpha}^i \mid i = 1, \dots, \Lambda_{\beta\alpha}\}$ be the eigenvalues of $\Phi_{\beta\alpha}$. Then

$$\sum_{\beta} \sum_i f_{\beta\alpha}^i = 1 \quad (\text{A.11})$$

$$\text{Index } E = \sum_{\beta} e_{\beta} \cdot \sum_{\alpha} \sum_i \frac{1}{f_{\beta\alpha}^i}. \quad (\text{A.12})$$

Let us choose a set $\{w_{\alpha}\}$ of positive numbers. Then the inequality between arithmetic and harmonic means weighted by $\{w_{\alpha}\}$ yields

$$\frac{\sum_{\alpha i} \frac{1}{f_{\beta\alpha}^i}}{\sum_{\alpha} \Lambda_{\beta\alpha} w_{\alpha}} \geq \frac{\sum_{\alpha} \Lambda_{\beta\alpha} w_{\alpha}}{\sum_{\alpha} \sum_i f_{\beta\alpha}^i w_{\alpha}^2} \quad (\text{A.13})$$

implying the estimate

$$\text{Index } E \geq \sum_{\beta} e_{\beta} \cdot \frac{(\sum_{\alpha} \Lambda_{\beta\alpha} w_{\alpha})^2}{\sum_{\alpha} \sum_i f_{\beta\alpha}^i w_{\alpha}^2} \quad (\text{A.14})$$

valid for all sequences $\{w_{\alpha}\}$ of positive numbers. Equality holds here iff there exist numbers u_{β} such that $f_{\beta\alpha}^i = u_{\beta}/w_{\alpha}$ for all i .

Lemma A.2 *Let $E: B \rightarrow A$ be a faithful conditional expectation over the connected inclusion $A \subset B$ with inclusion matrix Λ . Then*

$$\|\text{Index } E\| \geq \|\Lambda\|^2 \quad (\text{A.15})$$

where $\|\cdot\|$ denotes L^2 -operator norm.

Proof: $\|\text{Index } E\| = \max_{\beta} \sum_{\alpha,i} \frac{1}{f_{\beta\alpha}^i} \geq \max_{\beta} \frac{(\sum_{\alpha} \Lambda_{\beta\alpha} w_{\alpha})^2}{\sum_{\alpha,i} f_{\beta\alpha}^i w_{\alpha}^2}$ for all choices of positive numbers $\{w_{\alpha}\}$. Using the identity

$$\sum_{\beta} \sum_{\alpha,i} f_{\beta\alpha}^i w_{\alpha}^2 = \sum_{\alpha} w_{\alpha}^2$$

we obtain

$$\begin{aligned} \|\text{Index } E\| &\geq \left\{ \max_{\beta} \frac{(\sum_{\alpha} \Lambda_{\beta\alpha} w_{\alpha})^2}{\sum_{\alpha,i} f_{\beta\alpha}^i w_{\alpha}^2} \right\} \sum_{\beta} \sum_{\alpha,i} f_{\beta\alpha}^i w_{\alpha}^2 \cdot \frac{1}{\sum_{\alpha} w_{\alpha}^2} \\ &\geq \sum_{\beta} \frac{(\sum_{\alpha} \Lambda_{\beta\alpha} w_{\alpha})^2}{\sum_{\alpha} w_{\alpha}^2} = \frac{\|\Lambda w\|^2}{\|w\|^2} \end{aligned}$$

for all vectors w with positive entries. Choosing w to be the Perron-Frobenius eigenvector of $\Lambda^t \Lambda$ we obtain the desired result. *Q.e.d.*

Now we turn to the special case of trace preserving conditional expectations. Let $\tau: B \rightarrow \mathbb{C}$ be a faithful trace with trace vector $\{t_{\beta}\}$, i.e. $\tau(e_{\beta}^{IJ}) = \delta^{IJ} t_{\beta}$ and $t_{\beta} > 0$. Then the τ -preserving conditional expectation is the unique $E: B \rightarrow A$ satisfying $\tau(ab) = \tau(aE(b))$ for $a \in A$, $b \in B$. The Φ matrices of this E can be computed to be

$$\Phi_{\beta\alpha}^{i'i} = \delta^{i'i} \frac{t_{\beta}}{s_{\alpha}} \quad (\text{A.16})$$

where $s_{\alpha} = \sum_{\beta} t_{\beta} \Lambda_{\beta\alpha}$ is the trace vector of $\tau|_A$. Inserting this into the general formula (A.10) gives

$$\text{Index } E = \sum_{\beta} e_{\beta} \cdot \frac{\sum_{\beta'} (\Lambda \Lambda^t)_{\beta\beta'} t_{\beta'}}{t_{\beta}}. \quad (\text{A.17})$$

Definition A.3 Let $A \subset B$ be a connected inclusion with inclusion matrix Λ . Then the trace $\tau_M: B \rightarrow \mathbb{C}$ is called the Markov trace for $A \subset B$ if its trace vector t is the Perron-Frobenius eigenvector of $\Lambda \Lambda^t$. To make it unique we require $\tau_M(1) = 1$, i.e. $\sum_{\beta} m_{\beta} t_{\beta} = 1$. The τ_M -preserving conditional expectation $E_M: B \rightarrow A$ is called the Markov conditional expectation.

If $A \subset B$ is not connected then let $\{z_{\nu}\}$ be the set of minimal idempotents in $\text{Center } A \cap \text{Center } B$ and let τ_{ν} be the Markov trace of $z_{\nu} A \subset z_{\nu} B$. Then the trace preserving conditional expectation associated to any trace τ on B the restriction of which to the connected components $z_{\nu} A \subset z_{\nu} B$ are nonzero multiples of τ_{ν} is a unique conditional expectation $E_M: B \rightarrow A$, called the Markov conditional expectation.

Now a quick look at formula (A.17) yields the following

Scholium A.4 If E is a trace preserving conditional expectation then

$$\text{Index } E \in \text{Center } A \cap \text{Center } B \iff E = E_M \quad (\text{A.18})$$

and the index of E_M saturates the bound (A.15), i.e. for connected inclusions $\text{Index } E_M = 1 \cdot \|\Lambda\|^2$.

Finally we study connected inclusions for which the Markov index takes its naïve value $\dim B / \dim A$. Notice that since $\Lambda n = m$, we have the general estimate

$$\|\Lambda\|^2 \geq \frac{\|\Lambda n\|^2}{\|n\|^2} = \frac{\dim B}{\dim A} . \quad (\text{A.19})$$

Lemma A.5 *Let $A \subset B$ be a connected inclusion such that the norm of the inclusion matrix satisfies $\|\Lambda\|^2 = \dim B / \dim A$. Then the Markov trace τ_M is the left regular trace on B and the Markov index $\|\Lambda\|^2$ is an integer. Hence $\dim B$ is divisible by $\dim A$.*

Proof: Since $\frac{\|\Lambda n\|^2}{\|n\|^2} = \|\Lambda\|^2$, n is the Perron-Frobenius eigenvector of $\Lambda^t \Lambda$. But then $m = \Lambda n$ is the Perron-Frobenius eigenvector of $\Lambda \Lambda^t$, thus

$$t_\beta = \frac{m_\beta}{\dim B} \quad s_\alpha = \frac{n_\alpha}{\dim A} \quad (\text{A.20})$$

and τ_M and $\tau_M|_A$ are the normalized regular traces on B and A , respectively. Now the dimension vectors m and n satisfy the equations

$$\Lambda n = m , \quad \Lambda^t m = n \cdot I \quad (\text{A.21})$$

where $I = \|\Lambda\|^2$, the Markov index. The 2nd equation implies that $n_\alpha I$ are integers therefore if l denotes the greatest common divisor of $\{n_\alpha \mid \alpha \in \text{Sec } A\}$ then $lI \in \mathbb{Z}$. Now the 1st equation implies that each m_β is divisible by l , too, hence $m' = \frac{m}{l}$ and $n' = \frac{n}{l}$ are also integer vectors and satisfy $\Lambda^t m' = n' \cdot I$. Therefore I is an integer. *Q.e.d.*

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